Probabilistic Models for Occurrences

Modelling probabilistic dependencies (causalities) and independencies between discrete events of a scene

X_i random variable *models uncertain propositions about a scene*

$X_i = a$ hypothesis

Decomposition of joint probabilities:

$P(X_1, X_2, X_3, \dots, X_N) = P(X_1 \mid X_2, X_3, \dots, X_N) \cdot P(X_2 \mid X_3, X_4, \dots, X_N) \cdot \dots \cdot P(X_{N-1} \mid X_N) \cdot P(X_N)$

Simplification in the case of statistical independence:

X independent of X_i

$P(X \mid X_1, \dots X_{i-1}, X_i, X_{i+1}, \dots, X_N) = P(X \mid X_1, \dots X_{i-1}, X_{i+1}, \dots, X_N)$

Joint probability of N variables may be simplified by ordering the variables according to their direct dependence (causality).

1

Independence Causes Complexity Reduction

Assume that all random variables X_n of the JPD $P(X_1, X_2, X_3, \dots, X_N)$ have a domain size K. Then a fully general JPD requires K^N entries.

Example: N = 20, K = 10 => 10^{20} entries must be specified!

If all random variables are statistically independent, we have $P(X_1, X_2, X_3, \dots, X_N) = P(X_1) \cdot P(X_2) \cdot \dots \cdot P(X_N)$ and only KN entries are required.

Exploiting independencies can greatly reduce the size of a probability table!

2

Conditional Independence

It is useful to determine direct influences Y_i on a random variable X, because given the Y_i, X is independent of other Variables Z_k "upstream" to the Y_i.

Let dom(X) be the domain of X, i.e. the set of possible values of X.

A random variable X is independent of Z given Y if for all $x_i \in$ dom(X), for all $y_j \in$ dom(Y), and for all $z_k \in$ dom(Z),

$$P(X=x_i \mid Y=y_j, Z=z_k) = P(X=x_i \mid Y=y_j)$$

<u>Example</u>: X=plate_in_view, Y=plate_on_table, Z=want_to_eat

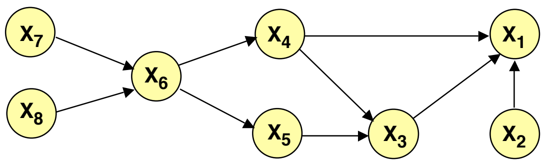
X Y Z	P(XYZ)	X Y Z	P(XYZ)
T T T	.096	F T T	.024
T T F	.064	F T F	.016
T F T	.0	F F T	.08
T F F	.0	F F F	.72

Check whether X is independent of Z given Y!

Causality Graph

Conditional dependencies (causality relations) of random variables define partial order.

Representation as a directed acyclic graph (DAG):



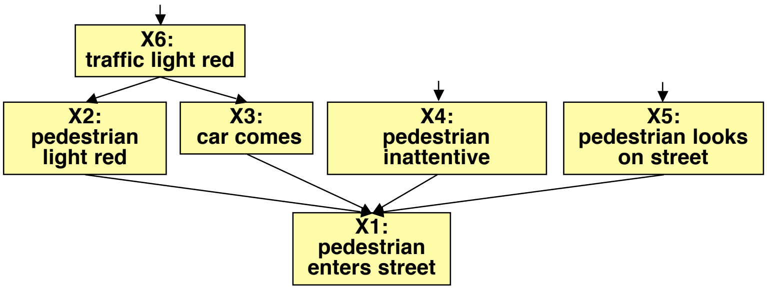
$P(X_1, X_2, X_3, \ldots, X_8) =$
$P(X_1 \mid X_2, X_3, X_4) \cdot P(X_2) \cdot P(X_3 \mid X_4, X_5) \cdot P(X_4 \mid X_6) \cdot P(X_5 \mid X_6) \cdot P(X_6 \mid X_7 X_8) \cdot P(X_7) \cdot P(X_8)$

For any DAG, we obtain the JPD as follows:

$Pa(X_i)$ parents of node X_i

$$P(X_1 \ldots X_N) = \prod_i P(X_i \mid Pa(X_i))$$

Example: Traffic Behaviour of Pedestrians



Conditional probability table for each node must be known

P(X1 I X2, X3, X4, X5)

X1	X2	X3	X4	X5	P
T	T	T	T	T	0.3
F	T	T	T	T	0.7
T	F	T	T	T	0.9
F	F	T	T	T	0.1
:	:	:	:	:	:

P(X2 I X6)

X2	X6	P
T	T	0.2
F	T	0.8
T	F	1.0
F	F	0.0

P(X3 I X6)

X3	X6	P
T	T	0.01
F	T	0.99
T	F	0.6
F	F	0.4

P(X4)

X4	P
T	0.1
F	0.9

P(X5)

X5	P
T	0.7
F	0.3

P(X6)

X6	P
T	0.7
F	0.3

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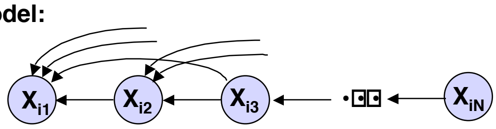
Bayes Nets are not Unique

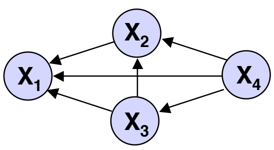
Using the chain rule, a JPD $P(X_1, X_2, \dots, X_N)$ may be expanded in N! ways:

$$P(X_1, X_2, \dots, X_N) = P(X_{i1} \mid X_{i2}, \dots, X_{iN}) \cdot P(X_{i2} \mid X_{i3}, \dots, X_{iN}) \cdot \dots \cdot P(X_{iN})$$

Even with no independencies, each chain rule expansion can be drawn as a graphical model:



<u>Example</u>:



Any JPD $P(X_1, X_2, X_3, X_4)$ can be represented by this Bayes Net.

For efficient inferences with a given JPD, it is important to find a Bayes Net with a low number of dependencies.

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Constructing a Bayes Net

By domain analysis:

1. Select discrete variables X_i relevant for domain
2. Establish partial order of variables according to causality
3. In the order of decreasing causality:
   (i) Generate node X_i in net
   (ii) As predecessors of X_i choose the smallest subset of nodes which are already in the net and from which X_i is causally dependent
   (iii) determine a table of conditional probabilities for X_i

By data analysis:

Use a learning method to establish a Bayes Net approximating the empirical joint probablity distribution.

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Computing Inferences

We want to use a Bayes Net for probabilistic inferences of the following kind:

Given a joint probability $P(X_1, \dots, X_N)$ represented by a Bayes Net, and evidence $X_{m_1}=a_{m_1}, \dots, X_{m_K}=a_{m_K}$ for some of the variables, what is the probability $P(X_n= a_i \mid X_{m_1}=a_{m_1}, \dots, X_{m_K}=a_{m_K})$ of an unobserved variable to take on a value a_i ?

In general this requires

- expressing a conditional probability by a quotient of joint probabilities

$$P(X_n= a_i \mid X_{m_1}=a_{m_1}, \dots, X_{m_K}=a_{m_K}) = \frac{P(X_n= a_i, X_{m_1}=a_{m_1}, \dots, X_{m_K}=a_{m_K})}{P(X_{m_1}=a_{m_1}, \dots, X_{m_K}=a_{m_K})}$$

- determining partial joint probabilities from the given total joint probability by summing out unwanted variables

$$P(X_{m_1}=a_{m_1}, \dots, X_{m_K}=a_{m_K}) = \sum_{x_{n_1}, \dots, x_{n_K}} P(X_{m_1}=a_{m_1}, \dots, X_{m_K}=a_{m_K}, X_{n_1}, \dots, X_{n_K})$$

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Normalization

Basic formula for computing the probability of a query variable X_n from a JPD $P(X_1, \dots, X_N)$ given evidence $X_{m_1}=a_{m_1}, \dots, X_{m_K}=a_{m_K}$:

$$P(X_n= a_i \mid X_{m_1}=a_{m_1}, \dots, X_{m_K}=a_{m_K}) = \frac{P(X_n= a_i, X_{m_1}=a_{m_1}, \dots, X_{m_K}=a_{m_K})}{P(X_{m_1}=a_{m_1}, \dots, X_{m_K}=a_{m_K})}$$

The denominator on the right is independent of a_i and constitutes a normalizing factor α. It can be computed by requiring that the conditional probabilities of all a_i sum to unity.

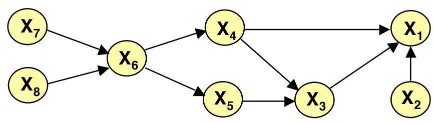
$$P(X_n= a_i \mid X_{m_1}=a_{m_1}, \dots, X_{m_K}=a_{m_K}) = \alpha \{ P(X_n= a_i, X_{m_1}=a_{m_1}, \dots, X_{m_K}=a_{m_K}) \}$$

Formulae are often written in this simplified form with α as a normalizing factor.

Factoring the JPD

JPDs can be computed from a Bayes Net more efficiently by ordering the "factors" so that only few summations and products must be computed.

<u>Example</u>:
Compute
$P(X_2=a, X_4=b \mid X_1=c, X_7 =d)$



$$P(X_2=a, X_4=b \mid X_1=c, X_7 =d) = \frac{P(X_2=a, X_4=b, X_1=c, X_7 =d)}{P(X_1=c, X_7 =d)}$$

$$P(X_2=a, X_4=b, X_1=c, X_7 =d) = \sum_{X_3} \sum_{X_5} \sum_{X_6} \sum_{X_8} P(X_1=c, X_2=a, X_3, X_4=b, X_5, X_6, X_7=d, X_8)$$

$$= \sum_{X_3} \sum_{X_5} \sum_{X_6} \sum_{X_8} P(X_1=c \mid X_2=a, X_3, X_4=b) \cdot P(X_2=a) \cdot P(X_3 \mid X_4=b, X_5) \cdot P(X_4 =b \mid X_6) \cdot P(X_5 \mid X_6) \cdot P(X_6 \mid X_7=d, X_8) \cdot P(X_7=d) \cdot P(X_8)$$

$$= P(X_2=a) \cdot P(X_7=d) \cdot \sum_{X_3} P(X_1=c \mid X_2=a, X_3, X_4=b) \cdot \sum_{X_5} P(X_3 \mid X_4=b, X_5) \cdot \sum_{X_6} P(X_4 =b \mid X_6) \cdot P(X_5 \mid X_6) \cdot \sum_{X_8} P(X_6 \mid X_7=d, X_8) \cdot P(X_8)$$

one possible order for efficient computation

Set-factoring Heuristic

Finding the <u>best</u> possible order for computing factors of a JPD is not tractable, in general. The set-factoring heuristic is a greedy (suboptimal) algorithm with often excellent results.

Given $\mathcal{X}$ set of random variables to be summed out

$\mathcal{F}$ set of factors to be combined

Set-factoring heuristic:

- Pick the pair of factors which produces the smallest probability table after combination and summing out as many variables of $\mathcal{X}$ as possible. Break ties by choosing the pair where most variables are summed out.
- Place resulting factor into set $\mathcal{F}$, remove summed-out variables from $\mathcal{X}$ and repeat procedure.

Example for Set-factoring Heuristic (1)

Compute

$$P(X_2=a,X_4=b,X_1=c,X_7=d) = \sum_{X_3}\sum_{X_5}\sum_{X_6}\sum_{X_8} P(X_1=c \mid X_2=a,X_3,X_4=b) \cdot P(X_2=a) \cdot P(X_3 \mid X_4=b,X_5) \cdot P(X_4=b \mid X_6) \cdot P(X_5 \mid X_6) \cdot P(X_6 \mid X_7=d,X_8) \cdot P(X_7=d) \cdot P(X_8)$$

<u>Step 1:</u> $\mathcal{X} = \{X_3, X_5, X_6, X_8\}$

$\mathcal{F} = \{P(X_1=c \mid X_2=a,X_3,X_4=b),\ P(X_2=a),\ P(X_3 \mid X_4=b,X_5),\ P(X_4=b \mid X_6),\ P(X_5 \mid X_6),\ P(X_6 \mid X_7=d,X_8),\ P(X_7=d),\ P(X_8)\}$

After extracting the constant factors $P(X_2=a)$ and $P(X_7=d)$, 6 factors remain, hence 15 possible pairs may be formed. Assuming equally sized domains, the set-factoring heuristic prefers 2 combinations:

(i) $P(X_1=c \mid X_2=a,X_3,X_4=b) \cdot P(X_3 \mid X_4=b,X_5)$ and summing out X_3
(ii) $P(X_6 \mid X_7=d,X_8) \cdot P(X_8)$ and summing out X_8

Choosing (ii), the new factor $P(X_6 \mid X_7=d)$ is computed and the sets are updated:

<u>Step 2:</u> $\mathcal{X} = \{X_3, X_5, X_6\}$

$\mathcal{F} = \{P(X_1=c \mid X_2=a,X_3,X_4=b),\ P(X_3 \mid X_4=b,X_5),\ P(X_4=b \mid X_6),\ P(X_5 \mid X_6),\ P(X_6 \mid X_7=d)\}$

Example for Set-factoring Heuristic (2)

The set-factoring heuristic prefers the combination:

$P(X_1{=}c \mid X_2{=}a, X_3, X_4{=}b) \cdot P(X_3 \mid X_4{=}b, X_5)$ and summing out X_3

The new factor $P(X_1{=}c \mid X_2{=}a, X_4{=}b, X_5)$ is computed and the sets are updated:

Step 3: $\mathcal{X} = \{X_5, X_6\}$

$\mathcal{F} = \{P(X_1{=}c \mid X_2{=}a, X_4{=}b, X_5),\ P(X_4{=}b \mid X_6),\ P(X_5 \mid X_6),\ P(X_6 \mid X_7{=}d)\}$

The set-factoring heuristic prefers the combination:

$P(X_1{=}c \mid X_2{=}a, X_4{=}b, X_5) \cdot P(X_5 \mid X_6)$ and summing out X_5

The new factor $P(X_1{=}c \mid X_2{=}a, X_4{=}b, X_6)$ is computed and the sets are updated:

Step 4: $\mathcal{X} = \{X_6\}$

$\mathcal{F} = \{P(X_1{=}c \mid X_2{=}a, X_4{=}b, X_6),\ P(X_4{=}b \mid X_6),\ P(X_6 \mid X_7{=}d)\}$

The set-factoring heuristic ranks all combinations equal. Choosing

$P(X_4{=}b \mid X_6) \cdot P(X_6 \mid X_7{=}d)$

we get the new factor $P(X_4{=}b, X_6 \mid X_7{=}d)$ and the updated sets:

Step 5: $\mathcal{X} = \{X_6\}$

$\mathcal{F} = \{P(X_1{=}c \mid X_2{=}a, X_4{=}b, X_6),\ P(X_4{=}b, X_6 \mid X_7{=}d)\}$

Example for Set-factoring Heuristic (3)

The final result follows from reassembling the summations outwards:

$$\begin{aligned}
P(X_2{=}a, X_4{=}b, X_1{=}c, X_7{=}d) = \\
& P(X_2{=}a) \cdot P(X_7{=}d) \\
& \cdot \sum_{X_6} P(X_4{=}b \mid X_6) \cdot P(X_6 \mid X_7{=}d) \\
& \cdot \sum_{X_5} P(X_5 \mid X_6) \\
& \cdot \sum_{X_3} P(X_1{=}c \mid X_2{=}a, X_3, X_4{=}b) \cdot P(X_3 \mid X_4{=}b, X_5) \\
& \cdot \sum_{X_8} P(X_6 \mid X_7{=}d, X_8) \cdot P(X_8)
\end{aligned}$$

If D is the size of the domains of the random variables, the number of multiplications is

$N_{mult} = D^2 + D^2 + D^2 + D$

This is the same as the number of multiplications for the manual ordering proposed earlier:

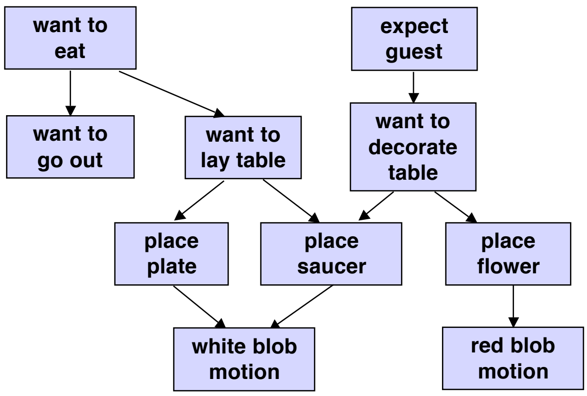
$N_{mult} = D^2 + D^2 + D^2 + D$

In this case, the heuristic did not reduce the computational expenses.

Dependance Analysis of Bayes Nets

The arcs in a Bayes Net indicate pairwise independence. Can one infer other independencies

- in general?
- given partial evidence in terms of node values?



<u>Example</u>:
If it is known

- that one wants to lay the table and
- that a white blob motion has been observed,

does this affect the probability of

- wanting to go out?
- red blob motion?

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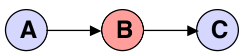
Blocking Evidence

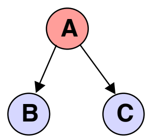
In general, (undirected) paths in a Bayes Net indicate possible flow of information. However, if hard evidence is given at an intermediate node, the path may be blocked.

Blocking situations:

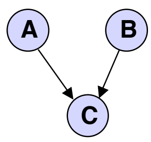
1. In a serial connection from A to C via B, evidence from A to C is blocked by hard evidence about B.



2. In a diverging connection from A to B and C, evidence from B to C is blocked by hard evidence about A.



3. In a converging situation from A and B to C, any evidence about C results in evidence transmitted between A and B.



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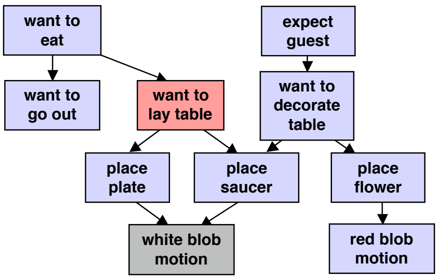
D-separation

"D-separation" = no flow of evidence from one node to another

Two nodes X and Y in a Bayes Net are d-separated if, for all paths between X and Y, there is an intermediate node Z for which either:

1. the connection is serial or diverging and the value of Z is known for certain; or

2. the connection is converging and neither Z (nor any of its descendants) have received any evidence at all.



Example:
Hard evidence for "want to lay table" blocks influence of evidence for "white blob motion" on "want to eat" and "want to go out", but not on any other nodes.

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Basic Kinds of Inferences

1. Causal reasoning, prediction

Given upstream evidence, ask for downstream probability

Example: Given "want to eat" is true, what is the probability of "white blob motion"?

2. Evidential reasoning, explanation

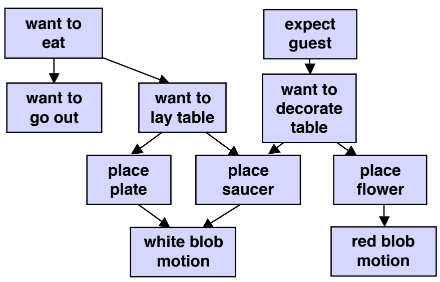
Given downstream evidence, ask for upstream probability

Example: Given "white blob motion" is true, what is the probability of "expect guest"?

3. Explaining away

Given evidence of a node with two parents and evidence for one of the parents, ask for probability of other parent node

Example: Given evidence for "place saucer" and "want to to eat", what is the probability of "want to decorate table"?



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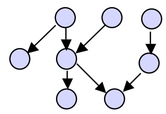
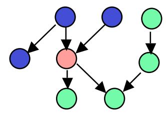
Evidence Propagation in Polytrees

polytree = DAG where each pair of distinct nodes is connected by a single (undirected) path



Any node X_k in a polytree separates the tree into an "upper" and "lower" part. Hence the marginal probability $P(X_k{=}c)$ can be computed from two factors.



$S^+ = \{X_i \text{ "above" } X_k\}$ $S^- = \{X_i \text{ below } X_k\}$

$$P(X_k{=}c) = \sum_{X_i \neq X_k} P(X_1 \ldots X_k{=}c \ldots X_N)$$

$$= \sum_{X_i \neq X_k} P(X_k{=}c \mid Pa(X_k)) \prod_{X_i \neq X_k} P(X_i \mid Pa(X_i)) \Big|_{X_k = c}$$

$$= \Big[\sum_{X_i \in S^+} P(X_k{=}c \mid Pa(X_k)) \prod P(X_i \mid Pa(X_i)) \Big] \cdot \Big[\sum_{X_i \in S^-} \prod P(X_i \mid Pa(X_i)) \Big|_{X_k = c} \Big]$$

$$= \pi(X_k{=}c) \cdot \lambda(X_k{=}c)$$ => propagation scheme is possible

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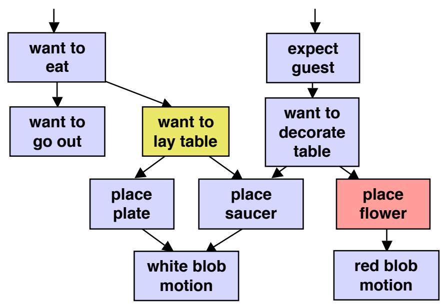
Approximate Inference in Bayesian Networks

- Inference in singly-connected Bayes Nets can be computed with O(N)
- Worst-case complexity in general Bayes Nets is exponential, hence approximate algorithms with less complexity are useful.

Basic idea:

Use random sampling (Monte Carlo method) to compute the approximate probability of an event based on a JPD and evidence.



Example: Determine P("place flower" | "want to lay table")

- Draw sample for each node based on probability conditioned on parent samples
- Repeat process many times
- Relative frequency of samples matching evidence converges to correct result in the limit.

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Sampling Methods

Recommended Reading:
Russell & Norvig: Artificial Intelligence - A Modern Approach, 2nd Ed., Prentice Hall, 2003

Direct Sampling:

Estimate the probability of an event without evidence by sampling a Bayes Net.

Rejection Sampling:

Estimate the probability of an event by sampling a Bayes Net and discarding all samples which do not match the evidence

Sampling with Likelihood Weighting:

Estimate the probability of an event by sampling a Bayes Net and weighting all samples according to their likelihood to generate the evidence

All three methods generate consistent estimates (which converge to the true value).

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Hidden Markov Models

A sequence of observations may be governed by underlying probabilistic state transitions.

Example: A person laying a table may plan to first place the plates, then the cups, then the cutlery in a cyclic order (with a chance to deviate from this order).

As usual in vision, observations may be disturbed and may provide uncertain evidence about the current state.

Such phenomena may be modelled by a Hidden Markov Model (HMM).

A (discrete) HMM is defined by

- a finite number of states $a_1, a_2, \ldots, a_K$
- a sequence of state transition events $t_0, t_1, \ldots, t_n$ (not necessarily times)
- probabilities of state transitions p_{ij} from state i to state j , each depending only on the previous state
- observations $b_1, b_2, \ldots, b_M$ probabilistically related to each state
- probabilities q_{km} which map states into observations

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Notation for HMM

- sequence of random variables $X^{(1)}, \dots, X^{(n)}$ (state variables) with values from $\{a_1, \dots, a_K\}$
- Markov Chain property of $X^{(1)}, \dots, X^{(n)}$: $P(X^{(n)}|X^{(n-1)} \dots X^{(1)}) = P(X^{(n)}|X^{(n-1)})$
  - if $P(X^{(n)}|X^{(n-1)})$ is independent of n, the Markov Chain is homogeneous
  - transition probabilities $P(X^{(n}=a_i|X^{(n-1)}=a_j)$ are represented by the state transition matrix

$$W^{(n)} = \begin{bmatrix} p_{11} & \cdots & p_{1K} \\ \vdots & & \\ p_{K1} & \cdots & p_{KK} \end{bmatrix}$$

- random variables $Y^{(1)}, \dots, Y^{(n)}$ (observations) with values from $\{b_1, \dots, b_M\}$
- observation probabilities $P(Y^{(n)}|X^{(n)})$ are represented by the matrix

$$Q = \begin{bmatrix} q_{11} & \cdots & q_{1M} \\ \vdots & & \\ q_{K1} & \cdots & q_{KM} \end{bmatrix}$$

- initial probabilities $□^T = [\ P(X^{(1)}=a_1)\ \ P(X^{(1)}=a_2) \dots P(X^{(1)}=a_K)\ ]$

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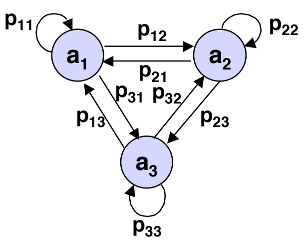
Properties of a Homogeneous HMM

Probability vector for state $X^{(2)}$: $□^{(2)} = W^T □$

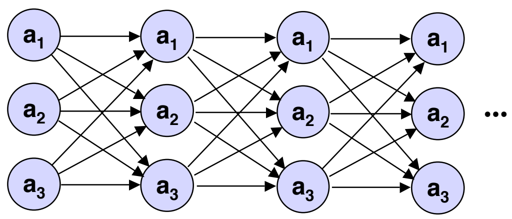
Probability vector for state $X^{(n)}$: $□^{(n)} = (W^T)^{n-1} □$

There is always a stationary distribution $□_s$ such that $□_s = W^T □_s$

Graphical representation:



Trellis ("Spalier") representation:



- each (directed) path corresponds to a legal sequence of states
- the probability of a path is equal to the product of the transition probabilities

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Paths through a HMM

Given a sequence of N observations, we want to find the most probable sequence of states which may have led to the observations.

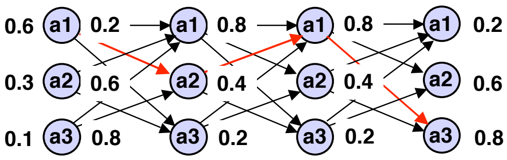
Extension of trellis representation

- arc weights leading into states $X^{(n)}$: transition probabilities p_{ij}
- node weights of states $X^{(n)}$: observation likelihoods q_{jm} for given observations $Y^{(n)} = b_{m_n}$
- product of initial probability and node and arc probabilities along path: $P(Y^{(1)}=b_{m_1}, \dots , Y^{(N)}=b_{m_N}, X^{(1)}=a_{k_1}, \dots , X^{(N)}=a_{k_N})$ probability of observations and states

Example:

$$W = \begin{bmatrix} 0.3 & 0.2 & 0.5 \\ 0.1 & 0.0 & 0.9 \\ 0.4 & 0.6 & 0.0 \end{bmatrix} \quad Q = \begin{bmatrix} 0.8 & 0.2 \\ 0.4 & 0.6 \\ 0.2 & 0.8 \end{bmatrix} \quad \square = \begin{bmatrix} 0.6 \\ 0.3 \\ 0.1 \end{bmatrix}$$

observations b_2, b_1, b_1, b_2



probability of observations along path are

$P(Y^{(1)}=b_2, Y^{(2)}=b_1, Y^{(3)}=b_1, Y^{(4)}=b_2$, states of path) =

$0.6 \cdot 0.2 \cdot 0.2 \cdot 0.4 \cdot 0.1 \cdot 0.8 \cdot 0.5 \cdot 0.8$

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Finding Most Probable Paths

The most probable sequence of states is found by maximizing

$$\max_{k_1 \dots k_N} P(X^{(1)}=a_{k_1}, \dots , X^{(N)}=a_{k_N} \mid Y^{(1)}=b_{m_1}, \dots , Y^{(N)}=b_{m_N}) = \max_{\underline{a}} P(\underline{a} \mid \underline{b})$$

Equivalently, the most probable sequence of states follows from

$$\max_{\underline{a}} P(\underline{a}\ \underline{b}) = \max_{\underline{a}} P(\underline{a} \mid \underline{b})\ P(\underline{b})$$

Hence the maximizing sequence of states can be found by exhaustive search of all path probabilities in the trellis. However, complexity is $O(K^N)$ with K = number of different states and N = length of sequence.

<u>The Viterbi Algorithm</u> does the job in O(KN)!

Overall maximization may be decomposed into a backward sequence of maximizations:

$$\max_{\underline{a}} P(\underline{a}\ \underline{b}) = \max_{k_1 \dots k_N} \square_{k_1} q_{k_1 m_1} \square_{n=2 \dots N} p_{k_{n-1}k_n} q_{k_{n-1}m_n}$$

$$= \max_{k_1} \square_{k_1} q_{k_1 m_1} (\max_{k_2} p_{k_1 k_2} q_{i_2 m_2} (\dots (\max_{k_N} p_{k_{N-1}k_N} q_{k_{N-1}m_N}) \dots))$$

Step N   Step N-1   Step 1

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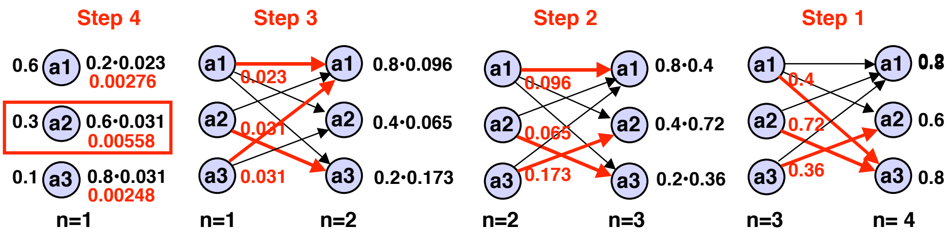
Example for Viterbi Algorithm

Typical maximization step of Viterbi algorithm:

$$\max_{k_n} \{ p_{k_{n-1}k_n} \cdot q_{k_{n-1}m_n} \cdot \text{<result of previous maximization step>} \}$$

Example as earlier:

$W = \begin{bmatrix} 0.3 & 0.2 & 0.5 \\ 0.1 & 0.0 & 0.9 \\ 0.4 & 0.6 & 0.0 \end{bmatrix}$ $Q = \begin{bmatrix} 0.8 & 0.2 \\ 0.4 & 0.6 \\ 0.2 & 0.8 \end{bmatrix}$ $\pi = \begin{bmatrix} 0.6 \\ 0.3 \\ 0.1 \end{bmatrix}$ observations b_2, b_1, b_1, b_2



red numbers show maximization results, red arrows maximizing transitions

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Model Evaluation for Given Observations

What is the likelihood that a particular HMM (out of several possible models) has generated the observations?

Likelihood of observations given model:

$$P(Y^{(1)}=b_{m_1}, \dots, Y^{(N)}=b_{m_N} \mid \text{model}) = P(\underline{b}) = \sum_{\underline{a}} P(\underline{a}\ \underline{b})$$

Instead of summing over all $\underline{a}$, one can use a forward algorithm based on the recursive formula:

$$\underline{P(a_j^{(n+1)}, b_{m_1}, \dots, b_{m_n}, b_{m_{n+1}})}$$
$$= P(a_j^{(n+1)}, b_{m_1}, \dots, b_{m_n}) \cdot P(b_{m_{n+1}} \mid a_j^{(n+1)})$$
$$= \sum_i [P(a_j^{(n+1)}, P(a_i^{(n)}, b_{m_1}, \dots, b_{m_n})] \cdot P(b_{m_{n+1}} \mid a_j^{(n+1)})$$
$$= \sum_i [P(a_j^{(n+1)} \mid P(a_i^{(n)}, b_{m_1}, \dots, b_{m_n}) P(a_i^{(n)}, b_{m_1}, \dots, b_{m_n})] \cdot P(b_{m_{n+1}} \mid a_j^{(n+1)})$$
$$= \sum_i [P(a_j^{(n+1)} \mid P(a_i^{(n)}) \cdot P(a_i^{(n)}, b_{m_1}, \dots, b_{m_n})] \cdot P(b_{m_{n+1}} \mid a_j^{(n+1)})$$
$$= \sum_i [p_{ij} \cdot \underline{P(a_i^{(n)}, b_{m_1}, \dots, b_{m_n})}] \cdot q_{j\, m_{n+1}}$$

Finally: $P(b_{m_1}, \dots, b_{m_N}) = \sum_i P(a_i^{(n+1)}, b_{m_1}, \dots, b_{m_N})$

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Example for Model Evaluation (1)

Computing the probability of observations stepwise as they come in.

Example as earlier:

$$W = \begin{bmatrix} 0.3 & 0.2 & 0.5 \\ 0.1 & 0.0 & 0.9 \\ 0.4 & 0.6 & 0.0 \end{bmatrix} \quad Q = \begin{bmatrix} 0.8 & 0.2 \\ 0.4 & 0.6 \\ 0.2 & 0.8 \end{bmatrix} \quad \pi = \begin{bmatrix} 0.6 \\ 0.3 \\ 0.1 \end{bmatrix}$$

observations
b_2, b_1, b_1, b_2

Step 1

$P(a_j^{(1)}, b_{m_1}) = \pi_j \cdot q_{j\, m_1}$

$P(a_1^{(1)}, b_2) = 0.6 \cdot 0.2 = 0.12$
$P(a_2^{(1)}, b_2) = 0.3 \cdot 0.6 = 0.18$
$P(a_3^{(1)}, b_2) = 0.1 \cdot 0.8 = 0.08$

Note that $P(b_{m_1}, \dots, b_{m_n})$ can be computed after each step by summing out the dependency on the state $X^{(n)}$.

Step 2

$P(a_j^{(2)}, b_{m_1}, b_{m_2}) = \sum [\, p_{ij} \cdot P(a_i^{(1)}, b_{m_1})] \cdot q_{j\, m_2}$

$P(a_1^{(2)}, b_2, b_1) = [\, 0.3 \cdot 0.12 + 0.1 \cdot 0.18 + 0.4 \cdot 0.08\,] \cdot 0.8 = 0.0314$
$P(a_2^{(2)}, b_2, b_1) = [\, 0.2 \cdot 0.12 + 0.6 \cdot 0.08\,] \cdot 0.4 = 0.0288$
$P(a_3^{(2)}, b_2, b_1) = [\, 0.5 \cdot 0.12 + 0.9 \cdot 0.18\,] \cdot 0.2 = 0.0072$

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Example for Model Evaluation (2)

Example continued:

$$W = \begin{bmatrix} 0.3 & 0.2 & 0.5 \\ 0.1 & 0.0 & 0.9 \\ 0.4 & 0.6 & 0.0 \end{bmatrix} \quad Q = \begin{bmatrix} 0.8 & 0.2 \\ 0.4 & 0.6 \\ 0.2 & 0.8 \end{bmatrix} \quad \pi = \begin{bmatrix} 0.6 \\ 0.3 \\ 0.1 \end{bmatrix}$$

observations
b_2, b_1, b_1, b_2

Step 3

$P(a_j^{(3)}, b_{m_1}, b_{m_2}, b_{m_3}) = \sum [\, p_{ij} \cdot P(a_j^{(2)}, b_{m_1}, b_{m_2})] \cdot q_{j\, m_3}$

$P(a_1^{(3)}, b_2, b_1, b_1) = [\, 0.3 \cdot 0.0314 + 0.1 \cdot 0.0288 + 0.4 \cdot 0.0072\,] \cdot 0.8 = 0.01214$
$P(a_2^{(3)}, b_2, b_1, b_1) = [\, 0.2 \cdot 0.0314 + 0.6 \cdot 0.0072\,] \cdot 0.4 = 0.00424$
$P(a_3^{(3)}, b_2, b_1, b_1) = [\, 0.5 \cdot 0.0314 + 0.9 \cdot 0.0288\,] \cdot 0.2 = 0.00832$

Step 4

$P(a_j^{(4)}, b_{m_1}, b_{m_2}, b_{m_3}, b_{m_4}) = \sum [\, p_{ij} \cdot P(a_j^{(2)}, b_{m_1}, b_{m_2}, b_{m_3})] \cdot q_{j\, m_4}$

$P(a_1^{(4)}, b_2, b_1, b_1, b_2) = [\, 0.3 \cdot 0.01214 + 0.1 \cdot 0.00424 + 0.4 \cdot 0.00832\,] \cdot 0.2 = 0.001479$
$P(a_2^{(4)}, b_2, b_1, b_1, b_2) = [\, 0.2 \cdot 0.01214 + 0.6 \cdot 0.00832\,] \cdot 0.6 = 0.004452$
$P(a_3^{(4)}, b_2, b_1, b_1, b_2) = [\, 0.5 \cdot 0.01214 + 0.9 \cdot 0.00424\,] \cdot 0.4 = 0.003954$

Final step

$P(b_{m_1}, b_{m_2}, b_{m_3}, b_{m_4}) = \sum P(a_j^{(4)}, b_{m_1}, b_{m_2}, b_{m_3}, b_{m_4}) = \boxed{0.009885}$

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