

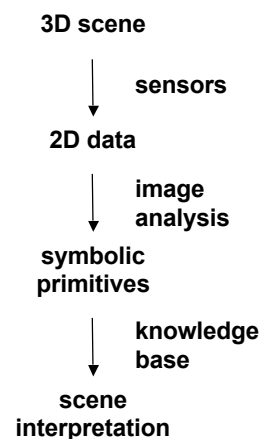
Probabilistic Models for Scene Interpretation

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Uncertainty in Scene Interpretation

Causes for uncertainty in scene interpretation:

- **Images give incomplete evidence for 3D scenes, allowing for multiple interpretations**
 - spatial and temporal clipping
 - occlusion
- **Image data may be corrupted by noise, image analysis will result in uncertain data**
- **Image analysis procedures may be coarse, allowing for multiple interpretations**
- **Models of the knowledge base may lack differentiation, allowing for multiple interpretations**
- **Logics of scene interpretation allow multiple interpretations**



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Probabilistic Approaches for Scene Interpretation

Several ways to use probabilistic representations for scene interpretation:

- Sensor modelling and sensor fusion
- Feature-based object classification
- Scene modelling (stationary + dynamic)
- Preference measure for logic-based interpretation

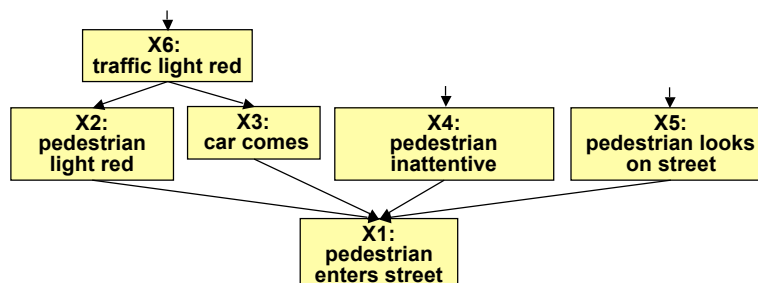
treated in
this course

Sage & Buxton 2005
http://www.ecvision.org/education/On-line_Cognitive_Vision_Course.htm

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Probabilistic Scene Modelling

- Random variables assigned to scene components:
 - events, occurrences
 - objects
 - properties
 - reified relations
- Probabilistic dependencies via joint distributions



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Random Variables and Chain Rule

Modelling probabilistic dependencies (causalities) and independencies between components of a scene

X_i random variable *models uncertain propositions about a scene*

$X_i = a$ hypothesis

Decomposition of joint probabilities:

$$P(X_1, X_2, X_3, \dots, X_N) = P(X_1 | X_2, X_3, \dots, X_N) \cdot P(X_2 | X_3, X_4, \dots, X_N) \cdot \dots \cdot P(X_{N-1} | X_N) \cdot P(X_N)$$

Simplification in the case of statistical independence:

X independent of X_i

$$P(X | X_1, \dots, X_{i-1}, X_i, X_{i+1}, \dots, X_N) = P(X | X_1, \dots, X_{i-1}, X_{i+1}, \dots, X_N)$$

Joint probability of N variables may be simplified by ordering the variables according to their direct dependence (causality).

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Independence Causes Complexity Reduction

Assume that all random variables X_n of the JPD $P(X_1, X_2, X_3, \dots, X_N)$ have a domain size K. Then a fully general JPD requires K^N entries.

Example: $N = 20, K = 10 \Rightarrow 10^{20}$ entries must be specified!

If all random variables are statistically independent, we have

$$P(X_1, X_2, X_3, \dots, X_N) = P(X_1) \cdot P(X_2) \cdot \dots \cdot P(X_N) \text{ and only } KN \text{ entries are required.}$$

Exploiting independencies can greatly reduce the size of a probability table!

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Conditional Independence

It is useful to determine direct influences Y_i on a random variable X , because given the Y_i , X is independent of other Variables Z_k "upstream" to the Y_i .

Let $\text{dom}(X)$ be the domain of X , i.e. the set of possible values of X .

A random variable X is independent of Z given Y if for all $x_i \in \text{dom}(X)$, for all $y_j \in \text{dom}(Y)$, and for all $z_k \in \text{dom}(Z)$,

$$P(X=x_i | Y=y_j, Z=z_k) = P(X=x_i | Y=y_j)$$

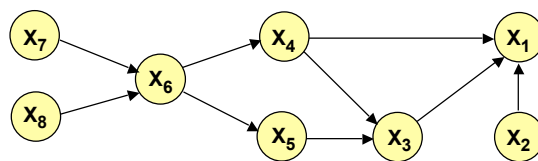
Example: $X=\text{plate_in_view}$, $Y=\text{plate_on_table}$, $Z=\text{want_to_eat}$

X Y Z	P(XYZ)	X Y Z	P(XYZ)	Check whether X is independent of Z given Y!
T T T	.096	F T T	.024	
T T F	.064	F T F	.016	
T F T	.0	F F T	.08	
T F F	.0	F F F	.72	

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Causality Graph

Conditional dependencies (causality relations) of random variables define partial order. Representation as a directed acyclic graph (DAG):



For each node X we need $P(X | \text{parents of } X)$ to obtain a specification of the JPD of all nodes.

$$P(X_1, X_2, X_3, \dots, X_8) = P(X_1 | X_2, X_3, X_4) \cdot P(X_2) \cdot P(X_3 | X_4, X_5) \cdot P(X_4 | X_6) \cdot P(X_5 | X_6) \cdot P(X_6 | X_7, X_8) \cdot P(X_7) \cdot P(X_8)$$

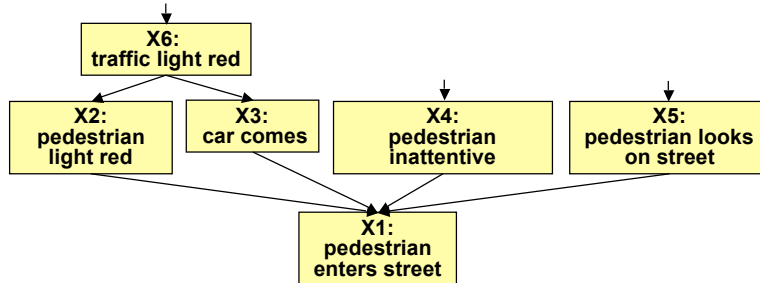
For any DAG, we obtain the JPD as follows:

$\text{Pa}(X_i)$ parents of node X_i

$$P(X_1 \dots X_N) = \prod_i P(X_i | \text{Pa}(X_i))$$

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Example: Traffic Behaviour of Pedestrians



Conditional probability table for each node must be known

P(X1 X2, X3, X4, X5)							P(X2 X6)			P(X3 X6)			P(X4)		P(X5)		P(X6)	
X1	X2	X3	X4	X5	P	X2	X6	P	X3	X6	P	X4	P	X5	P	X6	P	
T	T	T	T	T	0.3	T	T	0.2	T	T	0.01	T	0.1	T	0.7	T	0.7	
F	T	T	T	T	0.7	F	T	0.8	F	T	0.99	F	0.9	F	0.3	F	0.3	
T	F	T	T	T	0.9	T	F	1.0	T	F	0.6							
F	F	T	T	T	0.1	F	F	0.0	F	F	0.4							
⋮	⋮	⋮	⋮	⋮	⋮													

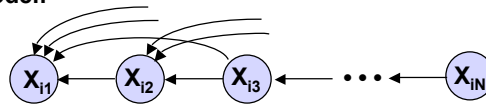
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Bayes Nets are not Unique

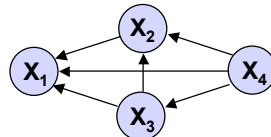
Using the chain rule, a JPD $P(X_1, X_2, \dots, X_N)$ may be expanded in $N!$ ways:

$$P(X_1, X_2, \dots, X_N) = P(X_{i_1} | X_{i_2}, \dots, X_{i_N}) \cdot P(X_{i_2} | X_{i_3}, \dots, X_{i_N}) \cdot \dots \cdot P(X_{i_N})$$

Even with no independencies, each chain rule expansion can be drawn as a graphical model:



Example:



Any JPD $P(X_1, X_2, X_3, X_4)$ can be represented by this Bayes Net.

For efficient inferences with a given JPD, it is important to find a Bayes Net with a low number of dependencies.

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Constructing a Bayes Net

By domain analysis:

1. Select discrete variables X_i relevant for domain
2. Establish partial order of variables according to causality
3. In the order of decreasing causality:
 - (i) Generate node X_i in net
 - (ii) As predecessors of X_i choose the smallest subset of nodes which are already in the net and from which X_i is causally dependent
 - (iii) determine a table of conditional probabilities for X_i

By data analysis:

Use a learning method to establish a Bayes Net approximating the empirical joint probability distribution.

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Computing Inferences

We want to use a Bayes Net for probabilistic inferences of the following kind:

Given a joint probability $P(X_1, \dots, X_N)$ represented by a Bayes Net, and evidence $X_{m_1}=a_{m_1}, \dots, X_{m_K}=a_{m_K}$ for some of the variables, what is the probability $P(X_n = a_i | X_{m_1}=a_{m_1}, \dots, X_{m_K}=a_{m_K})$ of an unobserved variable to take on a value a_i ?

In general this requires

- expressing a conditional probability by a quotient of joint probabilities

$$P(X_n = a_i | X_{m_1}=a_{m_1}, \dots, X_{m_K}=a_{m_K}) = \frac{P(X_n = a_i, X_{m_1}=a_{m_1}, \dots, X_{m_K}=a_{m_K})}{P(X_{m_1}=a_{m_1}, \dots, X_{m_K}=a_{m_K})}$$

- determining partial joint probabilities from the given total joint probability by summing out unwanted variables

$$P(X_{m_1}=a_{m_1}, \dots, X_{m_K}=a_{m_K}) = \sum_{X_{n_1}, \dots, X_{n_K}} P(X_{m_1}=a_{m_1}, \dots, X_{m_K}=a_{m_K}, X_{n_1}, \dots, X_{n_K})$$

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Normalization

Basic formula for computing the probability of a query variable X_n from a JPD $P(X_1, \dots, X_N)$ given evidence $X_{m_1}=a_{m_1}, \dots, X_{m_K}=a_{m_K}$:

$$P(X_n = a_i \mid X_{m_1}=a_{m_1}, \dots, X_{m_K}=a_{m_K}) = \frac{P(X_n = a_i, X_{m_1}=a_{m_1}, \dots, X_{m_K}=a_{m_K})}{P(X_{m_1}=a_{m_1}, \dots, X_{m_K}=a_{m_K})}$$

The denominator on the right is independent of a_i and constitutes a normalizing factor α . It can be computed by requiring that the conditional probabilities of all a_i sum to unity.

$$P(X_n = a_i \mid X_{m_1}=a_{m_1}, \dots, X_{m_K}=a_{m_K}) = \alpha \{ P(X_n = a_i, X_{m_1}=a_{m_1}, \dots, X_{m_K}=a_{m_K}) \}$$

Formulae are often written in this simplified form with α as a normalizing factor.

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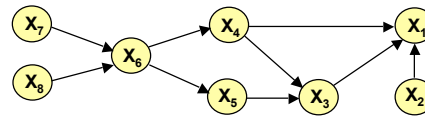
Factoring the JPD

JPDs can be computed from a Bayes Net more efficiently by ordering the "factors" so that only few summations and products must be computed.

Example:

Compute

$$P(X_2=a, X_4=b \mid X_1=c, X_7=d)$$



$$P(X_2=a, X_4=b \mid X_1=c, X_7=d) = \frac{P(X_2=a, X_4=b, X_1=c, X_7=d)}{P(X_1=c, X_7=d)}$$

$$\begin{aligned}
 P(X_2=a, X_4=b, X_1=c, X_7=d) &= \sum_{X_3} \sum_{X_5} \sum_{X_6} \sum_{X_8} P(X_1=c, X_2=a, X_3, X_4=b, X_5, X_6, X_7=d, X_8) \\
 &= \sum_{X_3} \sum_{X_5} \sum_{X_6} \sum_{X_8} P(X_1=c \mid X_2=a, X_3, X_4=b) \cdot P(X_2=a) \cdot P(X_3 \mid X_4=b, X_5) \cdot P(X_4=b \mid X_6) \cdot P(X_5 \mid X_6) \cdot \\
 &\quad \cdot P(X_6 \mid X_7=d, X_8) \cdot P(X_7=d) \cdot P(X_8) \\
 &= P(X_2=a) \cdot P(X_7=d) \cdot \sum_{X_3} P(X_1=c \mid X_2=a, X_3, X_4=b) \cdot \sum_{X_5} P(X_3 \mid X_4=b, X_5) \cdot \\
 &\quad \cdot \sum_{X_6} P(X_4=b \mid X_6) \cdot P(X_5 \mid X_6) \cdot \sum_{X_8} P(X_6 \mid X_7=d, X_8) \cdot P(X_8)
 \end{aligned}$$

one possible
order for
efficient
computation

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Set-factoring Heuristic

Finding the best possible order for computing factors of a JPD is not tractable, in general. The set-factoring heuristic is a greedy (suboptimal) algorithm with often excellent results.

Given \mathcal{X} set of random variables to be summed out

\mathcal{F} set of factors to be combined

Set-factoring heuristic:

- Pick the pair of factors which produces the smallest probability table after combination and summing out as many variables of \mathcal{X} as possible. Break ties by choosing the pair where most variables are summed out.
- Place resulting factor into set \mathcal{F} , remove summed-out variables from \mathcal{X} and repeat procedure.

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Example for Set-factoring Heuristic (1)

Compute

$$P(X_2=a, X_4=b, X_1=c, X_7=d) = \sum_{X_3} \sum_{X_5} \sum_{X_6} \sum_{X_8} P(X_1=c | X_2=a, X_3, X_4=b) \cdot P(X_2=a) \cdot P(X_3 | X_4=b, X_5) \cdot P(X_4=b | X_6) \cdot P(X_5 | X_6) \cdot P(X_6 | X_7=d, X_8) \cdot P(X_7=d) \cdot P(X_8)$$

Step 1: $\mathcal{X} = \{X_3, X_5, X_6, X_8\}$

$$\mathcal{F} = \{P(X_1=c | X_2=a, X_3, X_4=b), P(X_2=a), P(X_3 | X_4=b, X_5), P(X_4=b | X_6), P(X_5 | X_6), P(X_6 | X_7=d, X_8), P(X_7=d), P(X_8)\}$$

After extracting the constant factors $P(X_2=a)$ and $P(X_7=d)$, 6 factors remain, hence 15 possible pairs may be formed. Assuming equally sized domains, the set-factoring heuristic prefers 2 combinations:

- (i) $P(X_1=c | X_2=a, X_3, X_4=b) \cdot P(X_3 | X_4=b, X_5)$ and summing out X_3
- (ii) $P(X_6 | X_7=d, X_8) \cdot P(X_8)$ and summing out X_8

Choosing (ii), the new factor $P(X_6 | X_7=d)$ is computed and the sets are updated:

Step 2: $\mathcal{X} = \{X_3, X_5, X_6\}$

$$\mathcal{F} = \{P(X_1=c | X_2=a, X_3, X_4=b), P(X_3 | X_4=b, X_5), P(X_4=b | X_6), P(X_5 | X_6), P(X_6 | X_7=d)\}$$

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Example for Set-factoring Heuristic (2)

The set-factoring heuristic prefers the combination:

$$P(X_1=c | X_2=a, X_3, X_4=b) \cdot P(X_3 | X_4=b, X_5) \text{ and summing out } X_3$$

The new factor $P(X_1=c | X_2=a, X_4=b, X_5)$ is computed and the sets are updated:

Step 3: $\mathcal{X}_c = \{X_5, X_6\}$

$$\mathcal{F} = \{P(X_1=c | X_2=a, X_4=b, X_5), P(X_4=b | X_6), P(X_5 | X_6), P(X_6 | X_7=d)\}$$

The set-factoring heuristic prefers the combination:

$$P(X_1=c | X_2=a, X_4=b, X_5) \cdot P(X_5 | X_6) \text{ and summing out } X_5$$

The new factor $P(X_1=c | X_2=a, X_4=b, X_6)$ is computed and the sets are updated:

Step 4: $\mathcal{X}_c = \{X_6\}$

$$\mathcal{F} = \{P(X_1=c | X_2=a, X_4=b, X_6), P(X_4=b | X_6), P(X_6 | X_7=d)\}$$

The set-factoring heuristic ranks all combinations equal. Choosing

$$P(X_4=b | X_6) \cdot P(X_6 | X_7=d)$$

we get the new factor $P(X_4=b, X_6 | X_7=d)$ and the updated sets:

Step 5: $\mathcal{X}_c = \{X_6\}$

$$\mathcal{F} = \{P(X_1=c | X_2=a, X_4=b, X_6), P(X_4=b, X_6 | X_7=d)\}$$

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Example for Set-factoring Heuristic (3)

The final result follows from reassembling the summations outwards:

$$P(X_2=a, X_4=b, X_1=c, X_7=d) =$$

$$P(X_2=a) \cdot P(X_7=d)$$

$$\cdot \left[\sum_{X_6} P(X_4=b | X_6) \cdot P(X_6 | X_7=d) \right]$$

$$\cdot \left[\sum_{X_5} P(X_5 | X_6) \right]$$

$$\cdot \left[\sum_{X_3} P(X_1=c | X_2=a, X_3, X_4=b) \cdot P(X_3 | X_4=b, X_5) \right]$$

$$\cdot \left[\sum_{X_8} P(X_6 | X_7=d, X_8) \cdot P(X_8) \right]]]]]$$

If D is the size of the domains of the random variables, the number of multiplications is

$$N_{\text{mult}} = D^2 + D^3 + D^2 + D$$

This is more than the number of multiplications for the manual ordering proposed earlier:

$$N_{\text{mult}} = D^2 + D^2 + D^2 + D$$

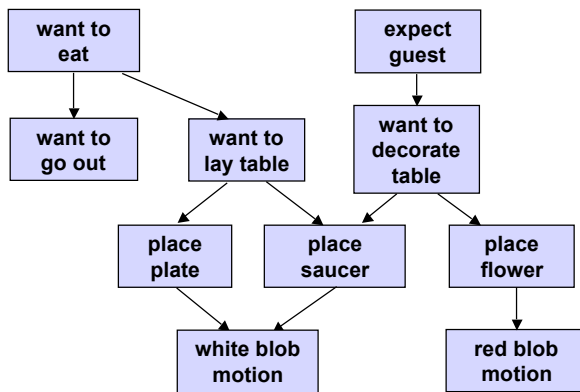
Obviously, the heuristic was not optimal in this case.

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Dependance Analysis of Bayes Nets

The arcs in a Bayes Net indicate pairwise independence. Can one infer other independencies

- in general?
- given partial evidence in terms of node values?



Example:

Given that a white blob motion has been observed, does this affect the probability of

- wanting to go out?
- red blob motion?

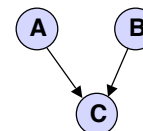
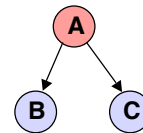
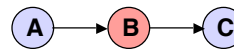
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Blocking Evidence

In general, (undirected) paths in a Bayes Net indicate possible flow of information. However, if hard evidence is given at an intermediate node, the path may be blocked.

Blocking situations:

1. In a serial connection from A to C via B, evidence from A to C is blocked by hard evidence about B.
2. In a diverging connection from A to B and C, evidence from B to C is blocked by hard evidence about A.
3. In a converging situation from A and B to C, any evidence about C results in evidence transmitted between A and B.



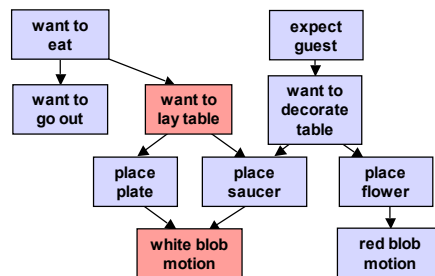
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D-separation

"D-separation" = no flow of evidence from one node to another

Two nodes X and Y in a Bayes Net are d-separated if, for all paths between X and Y, there is an intermediate node Z for which either:

1. the connection is serial or diverging and the value of Z is known for certain; or
2. the connection is converging and neither Z (nor any of its descendants) have received any evidence at all.



Example:

Hard evidence for "want to lay table" blocks influence of evidence for "white blob motion" on "want to eat" and "want to go out", but not on any other nodes.

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Basic Kinds of Inferences

1. Causal reasoning, prediction

Given upstream evidence, ask for downstream probability

Example: Given "want to eat" is true, what is the probability of "white blob motion"?

2. Evidential reasoning, explanation

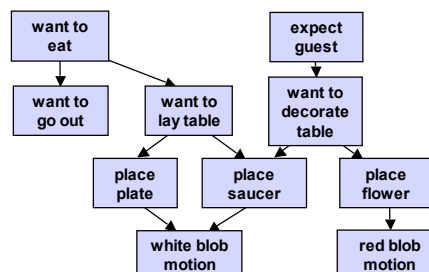
Given downstream evidence, ask for upstream probability

Example: Given "white blob motion" is true, what is the probability of "expect guest"?

3. Explaining away

Given evidence of a node with two parents and evidence for one of the parents, ask for probability of other parent node

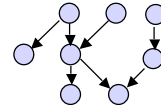
Example: Given evidence for "place saucer" and "want to eat", what is the probability of "want to decorate table"?



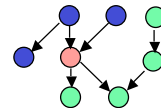
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Evidence Propagation in Polytrees

polytree = DAG where each pair of distinct nodes is connected by a single (undirected) path



Any node X_k in a polytree separates the tree into an "upper" and "lower" part. Hence the marginal probability $P(X_k=c)$ can be computed from two factors.



$S^+ = \{X_i \text{ above } X_k\}$ $S^- = \{X_i \text{ below } X_k\}$

$$\begin{aligned}
 P(X_k=c) &= \sum_{X_i \neq X_k} P(X_1 \dots X_k=c \dots X_N) \\
 &= \sum_{X_i \neq X_k} P(X_k=c \mid \text{Pa}(X_k)) \prod_{X_i \neq X_k} P(X_i \mid \text{Pa}(X_i)) \\
 &= \left[\sum_{X_i \in S^+} P(X_k=c \mid \text{Pa}(X_k)) \prod_{X_i \in S^+} P(X_i \mid \text{Pa}(X_i)) \right] \cdot \left[\sum_{X_i \in S^-} \prod_{X_k=c} P(X_i \mid \text{Pa}(X_i)) \right] \\
 &= \lambda(X_k=c) \cdot \rho(X_k=c) \quad \Rightarrow \text{propagation scheme is possible}
 \end{aligned}$$

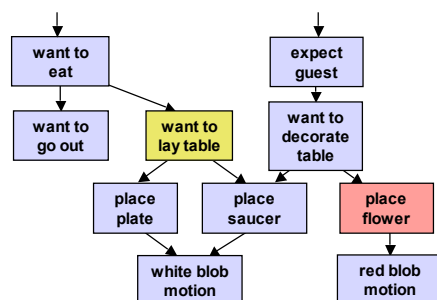
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Approximate Inference in Bayesian Networks

- Inference in singly-connected Bayes Nets can be computed with $O(N)$
- Worst-case complexity in general Bayes Nets is exponential, hence approximate algorithms with less complexity are useful.

Basic idea:

Use random sampling (Monte Carlo method) to compute the approximate probability of an event based on a JPD and evidence.



Example: Determine $P(\text{"place flower"} \mid \text{"want to lay table"})$

- Draw sample for each node based on probability conditioned on parent samples
- Repeat process many times
- Relative frequency of samples matching evidence converges to correct result in the limit.

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Sampling Methods

Direct Sampling:

Estimate the probability of an event without evidence by sampling a Bayes Net.

Recommended Reading:
Russell & Norvig: Artificial Intelligence - A Modern Approach, 2nd Ed., Prentice Hall, 2003

Rejection Sampling:

Estimate the probability of an event by sampling a Bayes Net and discarding all samples which do not match the evidence.

Sampling with Likelihood Weighting:

Estimate the probability of an event by sampling a Bayes Net and weighting all samples according to their likelihood to generate the evidence.

All three methods generate consistent estimates (which converge to the true value).

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Hidden Markov Models

A sequence of observations may be governed by underlying probabilistic state transitions.

- A person laying a table may plan to first place the plates, then the cups, then the cutlery in a cyclic order (with a chance to deviate from this order).
- Observations of a moving robot depend on its changing pose

As usual in vision, observations may be disturbed and may provide uncertain evidence about the current state.

Such phenomena may be modelled by a Hidden Markov Model (HMM).

A (discrete) HMM is defined by

- a finite number of states a_1, a_2, \dots, a_K
- a sequence of state transition events t_0, t_1, \dots, t_n (not necessarily times)
- probabilities of state transitions p_{ij} from state i to state j , each depending only on the previous state
- observations b_1, b_2, \dots, b_M probabilistically related to each state
- probabilities q_{km} which map states into observations

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Notation for HMM

- sequence of random variables $X^{(1)}, \dots, X^{(n)}$ (state variables) with values from $\{a_1, \dots, a_K\}$
- **Markov Chain property** of $X^{(1)}, \dots, X^{(n)}$: $P(X^{(n)}|X^{(n-1)} \dots X^{(1)}) = P(X^{(n)}|X^{(n-1)})$
 - if $P(X^{(n)}|X^{(n-1)})$ is independent of n , the Markov Chain is **homogeneous**
 - transition probabilities $P(X^{(n)}=a_j|X^{(n-1)}=a_i)$ are represented by the **state transition matrix**

$$W^{(n)} = \begin{bmatrix} p_{11} & \dots & p_{1K} \\ \vdots & & \vdots \\ p_{K1} & \dots & p_{KK} \end{bmatrix}$$

- random variables $Y^{(1)}, \dots, Y^{(n)}$ (observations) with values from $\{b_1, \dots, b_M\}$
- observation probabilities $P(Y^{(n)}|X^{(n)})$ are represented by the matrix

$$Q = \begin{bmatrix} q_{11} & \dots & q_{1M} \\ \vdots & & \vdots \\ q_{K1} & \dots & q_{KM} \end{bmatrix}$$

- initial probabilities $\underline{\pi}^T = [P(X^{(1)}=a_1) \ P(X^{(1)}=a_2) \ \dots \ P(X^{(1)}=a_K)]$

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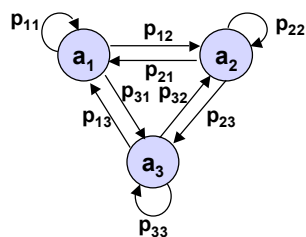
Properties of a Homogeneous HMM

Probability vector for state $X^{(2)}$: $\underline{\pi}^{(2)} = W^T \underline{\pi}$

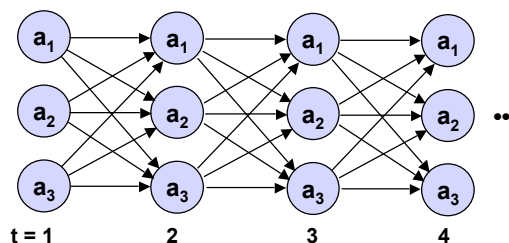
Probability vector for state $X^{(n)}$: $\underline{\pi}^{(n)} = (W^T)^{n-1} \underline{\pi}$

There is always a **stationary distribution** $\underline{\pi}_s$ such that $\underline{\pi}_s = W^T \underline{\pi}_s$

Graphical representation:



Trellis ("Spalier") representation:



- each (directed) path corresponds to a legal sequence of states
- the probability of a path is equal to the product of the transition probabilities

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Paths through a HMM

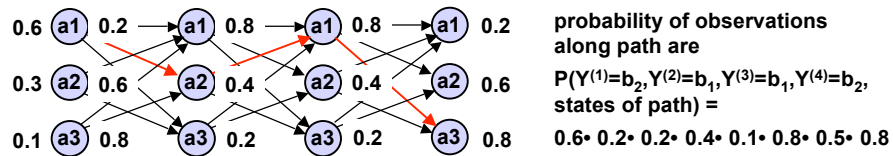
Given a sequence of N observations, we want to find the most probable sequence of states which may have led to the observations.

Extension of trellis representation

- arc weights leading into states $X^{(n)}$: \Rightarrow transition probabilities p_{ij}
- node weights of states $X^{(n)}$: \Rightarrow observation likelihoods q_{jm} for given observations $Y^{(n)} = b_{m_n}$
- product of initial probability and node and arc probabilities along path: $\Rightarrow P(Y^{(1)}=b_{m_1}, \dots, Y^{(N)}=b_{m_N}, X^{(1)}=a_{k_1}, \dots, X^{(N)}=a_{k_N})$ probability of observations and states

Example:

$$W = \begin{bmatrix} 0.3 & 0.2 & 0.5 \\ 0.1 & 0.0 & 0.9 \\ 0.4 & 0.6 & 0.0 \end{bmatrix} \quad Q = \begin{bmatrix} 0.8 & 0.2 \\ 0.4 & 0.6 \\ 0.2 & 0.8 \end{bmatrix} \quad \pi = \begin{bmatrix} 0.6 \\ 0.3 \\ 0.1 \end{bmatrix} \quad \text{observations } b_2, b_1, b_1, b_2$$



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Finding Most Probable Paths

The most probable sequence of states is found by maximizing

$$\max_{k_1 \dots k_N} P(X^{(1)}=a_{k_1}, \dots, X^{(N)}=a_{k_N} \mid Y^{(1)}=b_{m_1}, \dots, Y^{(N)}=b_{m_N}) = \max_{\underline{a}} P(\underline{a} \mid \underline{b})$$

Equivalently, the most probable sequence of states follows from

$$\max_{\underline{a}} P(\underline{a} \mid \underline{b}) = \max_{\underline{a}} P(\underline{a} \mid \underline{b}) P(\underline{b})$$

Hence the maximizing sequence of states can be found by exhaustive search of all path probabilities in the trellis. However, complexity is $O(K^N)$ with K = number of different states and N = length of sequence.

The Viterbi Algorithm does the job in $O(KN)$!

Overall maximization may be decomposed into a backward sequence of maximizations:

$$\begin{aligned} \max_{\underline{a}} P(\underline{a} \mid \underline{b}) &= \max_{k_1 \dots k_N} p_{k_1} q_{k_1 m_1} \prod_{n=2 \dots N} p_{k_{n-1} k_n} q_{k_{n-1} m_n} \\ &= \max_{k_1} p_{k_1} q_{k_1 m_1} (\max_{k_2} p_{k_1 k_2} q_{k_2 m_2} (\dots (\max_{k_N} p_{k_{N-1} k_N} q_{k_{N-1} m_N} \dots))) \end{aligned}$$

Step N Step N-1 ••• Step 1

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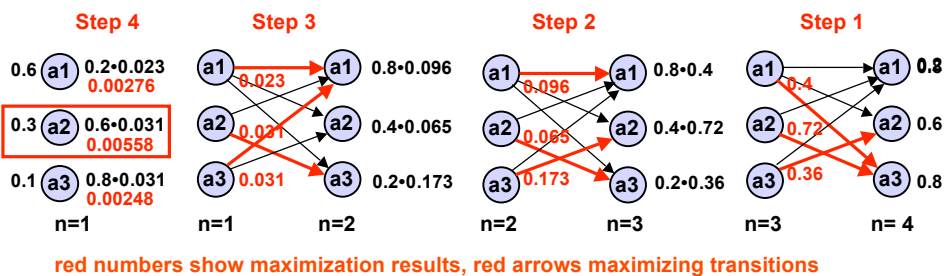
Example for Viterbi Algorithm

Typical maximization step of Viterbi algorithm:

$$\max_{k_n} \{ p_{k_{n-1}k_n} \cdot q_{k_{n-1}m_n} \cdot \text{<result of previous maximization step> } \}$$

Example as earlier:

$$W = \begin{bmatrix} 0.3 & 0.2 & 0.5 \\ 0.1 & 0.0 & 0.9 \\ 0.4 & 0.6 & 0.0 \end{bmatrix} \quad Q = \begin{bmatrix} 0.8 & 0.2 \\ 0.4 & 0.6 \\ 0.2 & 0.8 \end{bmatrix} \quad \pi = \begin{bmatrix} 0.6 \\ 0.3 \\ 0.1 \end{bmatrix} \quad \text{observations } b_2, b_1, b_1, b_2$$



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Model Evaluation for Given Observations

What is the likelihood that a particular HMM (out of several possible models) has generated the observations?

Likelihood of observations given model:

$$P(Y^{(1)}=b_{m_1}, \dots, Y^{(N)}=b_{m_N} \mid \text{model}) = P(\mathbf{b}) = \sum_{\mathbf{a}} P(\mathbf{a} \mid \mathbf{b})$$

Instead of summing over all \mathbf{a} , one can use a forward algorithm based on a recursive formula:

$$P(b_{m_1}, \dots, b_{m_N}) = \sum_i P(a_i^{(n+1)}, b_{m_1}, \dots, b_{m_N})$$

$$P(a_j^{(n+1)}, b_{m_1}, \dots, b_{m_n}, b_{m_{n+1}})$$

$$\begin{aligned} &= P(a_j^{(n+1)}, b_{m_1}, \dots, b_{m_n}) \cdot P(b_{m_{n+1}} \mid a_j^{(n+1)}) \\ &= \sum_i [P(a_j^{(n+1)}, a_i^{(n)}, b_{m_1}, \dots, b_{m_n})] \cdot P(b_{m_{n+1}} \mid a_j^{(n+1)}) \\ &= \sum_i [P(a_j^{(n+1)} \mid a_i^{(n)}, b_{m_1}, \dots, b_{m_n}) P(a_i^{(n)}, b_{m_1}, \dots, b_{m_n})] \cdot P(b_{m_{n+1}} \mid a_j^{(n+1)}) \\ &= \sum_i [P(a_j^{(n+1)} \mid a_i^{(n)}) \cdot P(a_i^{(n)}, b_{m_1}, \dots, b_{m_n})] \cdot P(b_{m_{n+1}} \mid a_j^{(n+1)}) \\ &= \sum_i [p_{ij} \cdot P(a_i^{(n)}, b_{m_1}, \dots, b_{m_n})] \cdot q_{j m_{n+1}} \end{aligned}$$

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Example for Model Evaluation (1)

Computing the probability of observations stepwise as they come in.

Example as earlier:

$$W = \begin{bmatrix} 0.3 & 0.2 & 0.5 \\ 0.1 & 0.0 & 0.9 \\ 0.4 & 0.6 & 0.0 \end{bmatrix} \quad Q = \begin{bmatrix} 0.8 & 0.2 \\ 0.4 & 0.6 \\ 0.2 & 0.8 \end{bmatrix} \quad \pi = \begin{bmatrix} 0.6 \\ 0.3 \\ 0.1 \end{bmatrix} \quad \begin{array}{l} \text{observations} \\ b_2, b_1, b_1, b_2 \end{array}$$

Step 1

$$P(a_j^{(1)}, b_{m_1}) = \pi_j \cdot q_{j m_1}$$

$$\begin{aligned} P(a_1^{(1)}, b_2) &= 0.6 \cdot 0.2 = 0.12 \\ P(a_2^{(1)}, b_2) &= 0.3 \cdot 0.6 = 0.18 \\ P(a_3^{(1)}, b_2) &= 0.1 \cdot 0.8 = 0.08 \end{aligned}$$

Note that $P(b_{m_1}, \dots, b_{m_p})$ can be computed after each step by summing out the dependency on the state $X^{(n)}$.

Step 2

$$P(a_j^{(2)}, b_{m_1}, b_{m_2}) = \sum_i [p_{ij} \cdot P(a_i^{(1)}, b_{m_1})] \cdot q_{j m_2}$$

$$\begin{aligned} P(a_1^{(2)}, b_2, b_1) &= [0.3 \cdot 0.12 + 0.1 \cdot 0.18 + 0.4 \cdot 0.08] \cdot 0.8 = 0.0314 \\ P(a_2^{(2)}, b_2, b_1) &= [0.2 \cdot 0.12 + 0.6 \cdot 0.08] \cdot 0.4 = 0.0288 \\ P(a_3^{(2)}, b_2, b_1) &= [0.5 \cdot 0.12 + 0.9 \cdot 0.18] \cdot 0.2 = 0.0072 \end{aligned}$$

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Example for Model Evaluation (2)

Example continued:

$$W = \begin{bmatrix} 0.3 & 0.2 & 0.5 \\ 0.1 & 0.0 & 0.9 \\ 0.4 & 0.6 & 0.0 \end{bmatrix} \quad Q = \begin{bmatrix} 0.8 & 0.2 \\ 0.4 & 0.6 \\ 0.2 & 0.8 \end{bmatrix} \quad \pi = \begin{bmatrix} 0.6 \\ 0.3 \\ 0.1 \end{bmatrix} \quad \begin{array}{l} \text{observations} \\ b_2, b_1, b_1, b_2 \end{array}$$

Step 3

$$P(a_j^{(3)}, b_{m_1}, b_{m_2}, b_{m_3}) = \sum_i [p_{ij} \cdot P(a_i^{(2)}, b_{m_1}, b_{m_2})] \cdot q_{j m_3}$$

$$\begin{aligned} P(a_1^{(3)}, b_2, b_1, b_1) &= [0.3 \cdot 0.0314 + 0.1 \cdot 0.0288 + 0.4 \cdot 0.0072] \cdot 0.8 = 0.01214 \\ P(a_2^{(3)}, b_2, b_1, b_1) &= [0.2 \cdot 0.0314 + 0.6 \cdot 0.0072] \cdot 0.4 = 0.00424 \\ P(a_3^{(3)}, b_2, b_1, b_1) &= [0.5 \cdot 0.0314 + 0.9 \cdot 0.0288] \cdot 0.2 = 0.00832 \end{aligned}$$

Step 4

$$P(a_j^{(4)}, b_{m_1}, b_{m_2}, b_{m_3}, b_{m_4}) = \sum_i [p_{ij} \cdot P(a_i^{(3)}, b_{m_1}, b_{m_2}, b_{m_3})] \cdot q_{j m_4}$$

$$\begin{aligned} P(a_1^{(4)}, b_2, b_1, b_1, b_2) &= [0.3 \cdot 0.01214 + 0.1 \cdot 0.00424 + 0.4 \cdot 0.00832] \cdot 0.2 = 0.001479 \\ P(a_2^{(4)}, b_2, b_1, b_1, b_2) &= [0.2 \cdot 0.01214 + 0.6 \cdot 0.00832] \cdot 0.6 = 0.004452 \\ P(a_3^{(4)}, b_2, b_1, b_1, b_2) &= [0.5 \cdot 0.01214 + 0.9 \cdot 0.00424] \cdot 0.4 = 0.003954 \end{aligned}$$

Final step

$$P(b_{m_1}, b_{m_2}, b_{m_3}, b_{m_4}) = \sum_j P(a_j^{(4)}, b_{m_1}, b_{m_2}, b_{m_3}, b_{m_4}) = \boxed{0.009885}$$

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