Shape Preserving Digitization of Ideal and Blurred Binary Images

Ullrich Köthe and Peer Stelldinger

Cognitive Systems Group, University of Hamburg,
Vogt-Köln-Str. 30, D-22527 Hamburg, Germany

Abstract. In order to make image analysis methods more reliable it is important to analyse to what extend shape information is preserved during image digitization. Most existing approaches to this problem consider topology preservation and are restricted to ideal binary images. We extend these results in two ways. First, we characterize the set of binary images which can be correctly digitized by both regular and irregular sampling grids, such that not only topology is preserved but also the Hausdorff distance between the original image and the reconstruction is bounded. Second, we prove an analogous theorem for gray scale images that arise from blurring of binary images with a certain filter type. These results are steps towards a theory of shape digitization applicable to real optical systems.

1 Introduction

When an analog image is digitized, much of its information may get lost. Therefore, it is important to understand which information is preserved. In this paper, we will be concerned with the problem of *shape* preservation. In particular, we would like discrete regions to have the same topology as their analog originals, and geometric distortions to be bounded. This problem of topology preservation was first investigated by Pavlidis [3]. He showed that a particular class of binary analog shapes (which we will call r-regular shapes, cf. definition 4) does not change topology under discretization with any sufficiently dense square grid. Similarly, Serra showed in [5] that the homotopy tree of r-regular sets is preserved under discretization with any sufficiently dense hexagonal grid. Both results apply to binary sets and the so called *subset digitization*, where a pixel is considered part of the digital shape iff its center is element of the given set.

Real images are always subjected to a certain amount of blurring before digitization. Blurring is an unavoidable property of any real optical system. It can be described by a convolution of the analog image with the *point spread function* (PSF) of the optical system. After convolution, analog images are no longer binary, and the above theorems do not apply. Latecki et al. [1] theorefore generalized the findings of Pavlidis to other digitizations including the *square subset* and *intersection* digitizations. These digitizations can be interpreted as subset digitizations of a *level set* of the blurred image where the PSF is a square

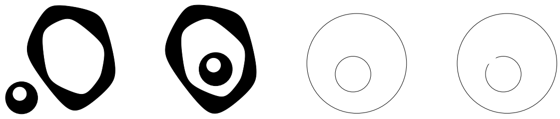


Fig. 1. Comparison of similarity criteria. (a) and (b) are topologically equivalent, (b) and (c) have the same homotopy tree, (c) and (d) have a very small Hausdorff distance when overlaid. No pair fulfills more than one condition.

with the same size as the pixels. Under this paradigm, topology preservation requires to halve the sampling distance.

In contrast, Ronse and Tajine [4] based their approach to digitization on the Hausdorff distance, i.e. a geometric measure of shape similarity. They proved that in the limit of infinitely dense sampling the Hausdorff distance between the original and digitized shapes converges to zero. However, they do not analyse under which circumstances the topology remains unchanged.

In this paper, we combine the three shape similarity criteria topological equivalence, identical homotopy tree and bounded Hausdorff distance. We prove that r-regularity is a sufficient condition for an analog set to be reconstructible (in the sense that all three criteria are met simultaneously) by any regular or irregular grid with sampling distance smaller than r. The results of [3,5] are obtained as corollaries of this theorem. We also apply these findings to binary images blurred with a flat disk-like PSF and show that the sampling density has to be increased according to the PSF's radius to ensure correct reconstruction.

2 Shape Similarity

Given two sets A and B, their similarity can be expressed in several ways. The most fundamental is topological equivalence. A and B are topologically equivalent if there exists a bijective function $f : A \to B$ with f and f^{-1} continuous. Such a function is called a homeomorphism. However, it does not completely characterize the topology of a set when it is embedded in the plane $\mathbb{R}^2$. Therefore, [5] introduced the homotopy tree which encodes whether some components of A enclose others in a given embedding. Fig. 1 (a) to (c) illustrate how shapes may differ if they are either topologically equivalent or have the same homotopy tree. We can capture both notions simultaneously when we extend the homeomorphism f to the entire $\mathbb{R}^2$ plane. Then it refers to a particular planar embedding of A and B and defines a mapping $A^c \to B^c$ for the set complements as well. This ensures preservation of both the topology and the homotopy tree. We call this an $\mathbb{R}^2$-*homeomorphism.*

Geometric similarity between two shapes can be measured by the *Hausdorff distance*

$$d_H(\partial A, \partial B) = \max\left(\max_{\boldsymbol{x}\in\partial A}\min_{\boldsymbol{y}\in\partial B} d(\boldsymbol{x},\boldsymbol{y}), \max_{\boldsymbol{y}\in\partial B}\min_{\boldsymbol{x}\in\partial A} d(\boldsymbol{x},\boldsymbol{y})\right)$$

between the shapes' boundaries. Fig. 1 (c) and (d) shows two shapes with small Hausdorff distance that are not $\mathbb{R}^2$-topologically equivalent.

All these criteria are necessary to regard a reconstructed image as similar to the original. Thus we combine them and call two sets *r-similar* if there exists a $\mathbb{R}^2$-homeomorphism that maps A into B, and $d_H(\partial A, \partial B) \le r$. That is, two sets A, B are r-similar, iff they are topologically equivalent, have the same homotopy tree, and their boundaries have a bounded Hausdorff distance.

3 Reconstructible Images

A set $A \subseteq \mathbb{R}^2$ can be transformed into an *analog binary image* by means of the characteristic function of the set $\chi_A : \mathbb{R}^2 \to \{0,1\}$, $\chi_A(\boldsymbol{x}) = 1$ iff $\boldsymbol{x} \in A$. A discretisation is obtained by storing the values of this image only at a countable number of sampling points. To characterize sampling formally, we must restrict the distance of the sampling points:

Definition 1. *A countable set $S \subset \mathbb{R}^2$ of points with $d_H(\mathbb{R}^2, S) \le r$ for some $r \in \mathbb{R}_+$ such that for each bounded set A the subset $S \cap A$ is finite, is called* r-grid*. The elements of S are the* sampling points*, and their associated Euclidean Voronoi regions are the* pixels*:*

$$\mathrm{Pixel}_S : S \to \mathcal{P}(\mathbb{R}^2), \quad \mathrm{Pixel}_S(\boldsymbol{s}) := \{\boldsymbol{x} : \forall \boldsymbol{s'} \in S \setminus \{\boldsymbol{s}\} : |\boldsymbol{x} - \boldsymbol{s}| \le |\boldsymbol{x} - \boldsymbol{s'}|\}$$

The intersection of $A \subseteq \mathbb{R}^2$ with S is called the S-digitization of A*, and the restriction of the domain of A's characteristic function to S is the associated* digital binary image*:*

$$\mathrm{Dig}_S(A) := A \cap S$$
$$\mathrm{DigitalImage}_S(\chi_A) := \chi_A|_S : S \to \{0,1\}$$

This definition is very broad and captures not only the usual rectangular and square grids, but also other regular and even irregular grids, provided their Voronoi regions have bounded radius, see fig. 2.

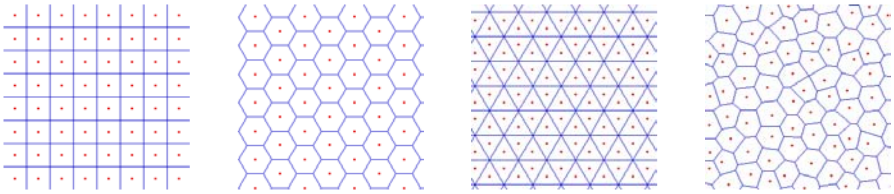
As it is not useful to directly compare a discrete set with an analog one, we reconstruct an analog set from the given digitization. This is done by assigning the information stored at each sampling point to the entire surrounding pixel:

Definition 2. *Given a set $A \subseteq \mathbb{R}^2$ and a grid S, the* S-reconstruction *of $\mathrm{Dig}_S(A)$ is defined as*

$$\hat{A} = \mathrm{Rec}_S(\mathrm{Dig}_S(A)) = \bigcup_{\boldsymbol{s} \in (S \cap A)} \mathrm{Pixel}_S(\boldsymbol{s})$$

The results of a reconstruction process will be considered correct if the reconstructed set $\hat{A}$ is sufficiently similar to the original set A. Formally, we get

Definition 3. *A set $A \subseteq \mathbb{R}^2$ is* reconstructible *by an r-grid S if the S-reconstruction $\hat{A}$ is r-similar to A.*



(a) (b) (c) (d)

Fig. 2. Many different grid types can be described when pixels are defined as the Voronoi regions of suitably located sampling points. These include regular grids like the square (a), hexagonal (b) and trigonal ones (c), and irregular grids (d) as found in natural image acquisition devices like the human eye.

This definition imposes stricter conditions on reconstruction than preservation of topology or homotopy trees as used by Pavlidis and Serra. Pavlidis gave a weaker bound for the Hausdorff distance and did not prove that the homotopy tree remains unchanged, while Serra didn't prove topology preservation. Corollary 1 shows that their geometric sampling theorems can be strengthened according to our requirements. We recall the definition of the type of shapes they looked at:

Definition 4. *A compact set $A \subset \mathbb{R}^2$ is called* r-regular *iff for each boundary point of A it is possible to find two osculating open balls of radius r, one lying entirely in A and the other lying entirely in A^c.*

In the following we will show that an r-regular set is reconstructible by any grid with sufficiently small pixel size, regardless of the grid structure. The following lemmas describe some prerequisites. We only formulate them for the foreground A, but their claims and proofs apply to the background A^c analogously.

Lemma 1. *Let A be an r-regular set and $\hat{A}$ the reconstruction of A by an r'-grid S, with $0 < r' < r$. Then two sampling points lying in different components of A cannot lie in the same component of $\hat{A}$.*

Proof. Since the Hausdorff distance of two components of A is at least $2r$ (cf. [1,2]), and the S-reconstruction of any component A' is a subset of the r'-dilation of A', the Hausdorff distance between two components of $\hat{A}$ is at least $2r - 2r' > 0$. Thus the reconstruction process cannot merge two components of A. □

Lemma 2. *Let A' be a component of an r-regular set A, S be an r'-grid, $0 < r' < r'' < r$. Further, let $A'_\ominus = (A' \ominus \overline{\mathcal{B}}_{r''})^0$ be the interior of the erosion of A' with a closed ball of radius r'', and $S_i := \{\boldsymbol{s} \in S : \mathrm{Pixel}(\boldsymbol{s}) \cap A'_\ominus \neq \emptyset\}$ the set of all sampling points whose pixels intersect $A'_\ominus$. Then at least one member of S_i is in A'.*

Proof. Since A is r-regular, every component A' contains at least one ball of radius r. The center $\boldsymbol{m}$ of such a ball lies in $A'_\ominus$. Let $\boldsymbol{s} \in S$ be a sampling point with $\boldsymbol{m} \in \mathrm{Pixel}(\boldsymbol{s})$. Then s is also element of S_i and the distance between $\boldsymbol{s}$ and $\boldsymbol{m}$ is at most $r' < r''$. Thus, $\boldsymbol{s}$ lies within A'. □

Lemma 3. *Let A, A', S and S_i be defined as in lemma 2. Then any pair of pixels with sampling points in S_i is connected by a chain of adjacent pixels whose sampling points are also in S_i. Pixels are adjacent if they have a common boundary edge (direct neighborhood).*

Proof. Every component A' of an r-regular set A is r-regular, too. Thus $A'_\ominus$ is an open, connected set. Now let $\boldsymbol{s}_1$ and $\boldsymbol{s}_2$ be sampling points in S_i. The interior of their pixels intersects $A'_\ominus$, and there exist two points $\boldsymbol{s}'_1$, $\boldsymbol{s}'_2$ lying in $(\text{Pixel}(\boldsymbol{s}_1))^0 \cap A'_\ominus$ and $(\text{Pixel}(\boldsymbol{s}_2))^0 \cap A'_\ominus$ respectively. $\boldsymbol{s}'_1$ and $\boldsymbol{s}'_2$ can be connected by a path in $A'_\ominus$ which, without loss of generality, does not intersect any pixel corner. The sampling points of all pixels intersecting this path are in S_i as well. The order in which the path enters those pixels defines a chain of adjacent pixels. □

Lemma 4. *Let A, A', S and S_i be defined as in lemma 2. Then each sampling point lying in A' is either a member of S_i or is connected to a member of S_i by a chain of adjacent pixels whose sampling points all lie in A'.*

Proof. Let $\boldsymbol{c}$ be any sampling point in A'. Then there exists a ball of radius r in A' such that $\boldsymbol{c}$ lies in the ball. Let $\boldsymbol{m} \in A'_\ominus$ be the center of the ball. The halfline starting at $\boldsymbol{c}$ and going through $\boldsymbol{m}$ crosses the boundary of the convex $\text{Pixel}(\boldsymbol{c})$ at exactly one point $\boldsymbol{c}'$. If $d(\boldsymbol{c}, \boldsymbol{m}) \leq d(\boldsymbol{c}, \boldsymbol{c}')$, the point $\boldsymbol{m}$ is part of $\text{Pixel}(\boldsymbol{c})$ and thus $\boldsymbol{c} \in S_i$. If $d(\boldsymbol{c}, \boldsymbol{m}) > d(\boldsymbol{c}, \boldsymbol{c}')$, let g be the line defined by the edge of $\text{Pixel}(\boldsymbol{c})$ going through $\boldsymbol{c}'$. If there are two such lines (i.e. if $\boldsymbol{c}'$ is a corner of $\text{Pixel}(\boldsymbol{c})$), one is chosen arbitrarily. Due to the definition of Voronoi regions the point $\boldsymbol{c}''$ constructed by mirroring $\boldsymbol{c}$ on g is a sampling point in S, and $\text{Pixel}(\boldsymbol{c}'')$ is adjacent to $\text{Pixel}(\boldsymbol{c})$. Since $c := d(\boldsymbol{c}', \boldsymbol{c}) = d(\boldsymbol{c}', \boldsymbol{c}'')$, the point $\boldsymbol{c}''$ always lies on the circle of radius c with center $\boldsymbol{c}'$. Among all points on this circle, $\boldsymbol{c}$ has the largest distance to $\boldsymbol{m}$, and in particular $d(\boldsymbol{m}, \boldsymbol{c}'') < d(\boldsymbol{m}, \boldsymbol{c})$. Thus, the sampling point $\boldsymbol{c}''$ lies in A', and is closer to $\boldsymbol{m}$ than $\boldsymbol{c}$. We can repeat this construction iteratively to obtain a sequence of adjacent pixels whose sampling points successively get closer to $\boldsymbol{m}$. Since there are only finitely many sampling points in A', one such pixel will eventually intersect $A'_\ominus$. □

Theorem 1 (sampling theorem for ideal binary images). *Let $r \in \mathbb{R}_+$ and A an r-regular set. Then A is reconstructible with* any *r'-grid S, $0 < r' < r$.*

Proof. Due to lemma 2 there is a mapping of the foreground components of A to the foreground components of $\hat{A}$. Lemma 1 states that this mapping is injective, and from lemmas 3 and 4 follows surjectivity. The same holds for the background components of A and $\hat{A}$. This implies a one-to-one mapping between the boundaries of A and $\hat{A}$. Due to lemma 4, both the foreground and background components of $\hat{A}$ are connected via direct pixel neighborhood. Therefore, their boundaries are Jordan curves. The same holds for the boundaries of A due to r-regularity. Consequently, an R^2-homeomorphism can be constructed, and A and $\hat{A}$ are R^2-topologically equivalent.

It remains to be shown that the Hausdorff distance between the boundaries of A and $\hat{A}$ is restricted. Suppose to the contrary that $\partial\hat{A}$ contains a point $\boldsymbol{s}$ whose distance from ∂A exceeds r'. Due to the definition of an r'-grid, the sampling points of all pixels containing $\boldsymbol{s}$ are located in a circle around $\boldsymbol{s}$ with radius r'. Under the supposition, this circle would either be completely inside or outside A, and the pixels were all either in $\hat{A}$ or $\hat{A}^c$. Thus, $\boldsymbol{s}$ could not be on $\partial\hat{A}'$ – contradiction. Therefore, the Hausdorff distance between ∂A and $\partial\hat{A}$ is at most r'. □

This *geometric sampling theorem* does not only apply to square or hexagonal grids, but also to irregular grids as can be found in the human retina, see fig. 2. Moreover, if a set is reconstructible by some grid S due to this theorem, this also holds for any translated and rotated copy of the grid. Moreover, it can be shown that r-regularity is not only a sufficient but also a nessessary condition for a set to be reconstructible. That is, if A is not r-regular for some r, there exists an r-grid S such that the S-reconstruction is not topologically equivalent to A. Due to space limitations, the proof of this claim had to be omitted. The sampling theorems of Serra and Pavlidis are corollaries of theorem 1:

Corollary 1. *Let $S_1 := h_1 \cdot \mathbb{Z}$ be the square grid with grid size (minimal sampling point distance) h_1. Then every r-regular set with $r > \frac{h_1}{\sqrt{2}}$ is reconstructible with S_1. Let S_2 be the hexagonal grid with grid size h_2. Then every r-regular set with $r > \frac{h_2}{\sqrt{3}}$ is reconstructible with S_2.*

4 Sampling of Blurred Images

In the previous section we worked exclusively with the subset digitization where a sampling point is set if it lies within the foreground region of the binary image. Unfortunately, this digitization scheme can never be realized in practice: Every real optical system blurs the binary image before the light reaches the optical sensors. The finite area of real sensors introduces additional blurring. Both effects can be described by a convolution of the ideal binary image with a suitable point spread function. Thus, the image actually observed is always a gray-scale image. A binary image can be recovered by considering a particular level set $L_l = \{\boldsymbol{x} \in \mathbb{R}^2 | \hat{f}(\boldsymbol{x}) \geq l\}$ of the blurred image $\hat{f}$, i.e. by thresholding. Since thresholding and digitization commute, we can apply thresholding first and then digitize the resulting level set by standard subset digitization. (This order facilitates the following proofs.) Now the question arises if and how we can bound the difference between the original set before blurring and the S-reconstruction of a level set of the blurred image. We first analyse the relationship between the original set and an analog level set, and then between the level set and its S-reconstruction.

In order to get definitive results, we restrict ourselves to a particular type of PSF, namely flat disks of radius p. Flat, disk-shaped PSFs have the advantage that the result of the convolution can be calculated by measuring the area of sets. In the sequel, A shall be an r-regular set and k_p a disk PSF with radius $p < r$. If $K_p(\boldsymbol{c})$ denotes the PSF's support region after translation to the point

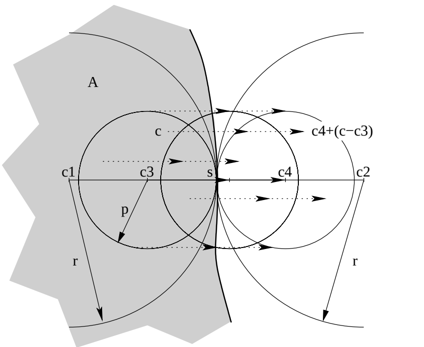


Fig. 3. If a p-ball is shifted orthogonally to the boundary ∂A from an inner osculating to an outer osculating position, its intersection area with A strictly decreases.

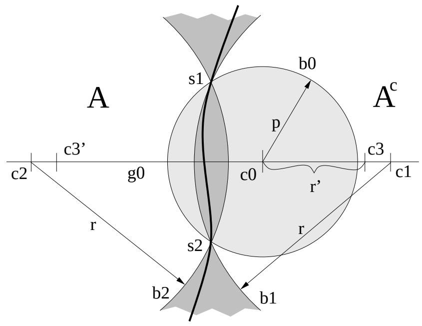


Fig. 4. The boundary of the circle b_0 centered at point $\boldsymbol{c}_0$ (light gray) intersects the boundary of the set A (bold line) at the two points $\boldsymbol{s}_1$ and $\boldsymbol{s}_2$. Since A is r-regular, its boundary can only lie within the area marked with dark gray.

$\boldsymbol{c}$, the result of the convolution at $\boldsymbol{c}$ is given by:

$$\hat{f}(\boldsymbol{c}) = (k_p \star \chi_A)(\boldsymbol{c}) = \frac{\|K_p(\boldsymbol{c}) \cap A\|}{\|K_p(\boldsymbol{c})\|}$$

where $\star$ denotes convolution and $\|.\|$ is the area size. Therefore, it is possible to derive properties of the level sets by purely geometrical means. Obviously, all interesting effects occur in a $2p$-wide strip $A_p = \partial A \oplus K_p$ around the boundary ∂A, because out of this strip the kernel does not overlap ∂A, and the gray values are either 0 or 1 there ($\oplus$ denotes morphological dilation). Level sets have the following property:

Lemma 5. *Let $\boldsymbol{s}$ be a point on ∂A, and let $\mathbf{c_1}$ and $\mathbf{c_2}$ be the centers of the inside and outside osculating circles of radius r. Moreover, let $\boldsymbol{c_3}$ and $\boldsymbol{c_4}$ be the two points on the normal $\overline{\mathbf{c_1 c_2}}$ with distance p from $\boldsymbol{s}$. Then the boundary of every level set has exactly one point in common with $\overline{\boldsymbol{c_3 c_4}}$.*

Proof. Consider a point $\boldsymbol{c}$ in $K_p(\boldsymbol{c_3})$ and translate the line segment $\overline{\boldsymbol{c_3 c_4}}$ by $\boldsymbol{c} - \boldsymbol{c}_3$ (see fig. 3). Because of the restricted curvature of ∂A, the translated line segment intersects ∂A at exactly one point. Thus, as $t \in [0, 1]$ increases, the area of $\|K_p(\boldsymbol{c_3} + t \cdot (\boldsymbol{c_4} - \boldsymbol{c_3})) \cap A\|$ is strictly decreasing. This area is proportional to the result of the convolution, so the same holds for the gray values. Since the p-ball centered in $\boldsymbol{c_3}$ is an inside osculating ball of A, the gray value at $\boldsymbol{c_3}$ is $f(0) = 1$. Likewise, $f(1) = 0$. This implies the lemma. $\square$

The curvature of the level set contours is bounded by the following lemma:

Lemma 6. *Let $\boldsymbol{c}_0 \in A_p$ be a point such that $(A \star K_p)(\boldsymbol{c}_0) = l$, $(0 < l < 1)$. Thus, $\boldsymbol{c}_0$ is part of level set L_l. Then there exists a circle b_{out} of radius $r_o \geq r' = r - p$*

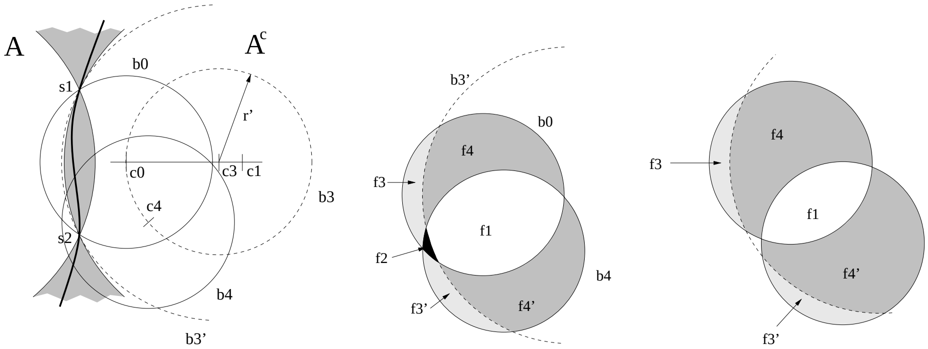


Fig. 5. Left: The gray level at any point $\boldsymbol{c}_4 \neq \boldsymbol{c}_0$ on b_3 is smaller than the gray level at $\boldsymbol{c}_0$; center and right: decomposition of the circles b_0 and b_4 into subsets (see text).

that touches $\boldsymbol{c}_0$ but is otherwise completely outside of L_l. Likewise, there is a circle b_{in} with radius $r_i \geq r'$ that is completely within L_l.

Proof. Consider the set $b_0 = K_p(\boldsymbol{c}_0)$ centered at $\boldsymbol{c}_0$. Let its boundary $\partial K_p(\boldsymbol{c}_0)$ intersect the boundary ∂A at the points $\boldsymbol{s}_1$ and $\boldsymbol{s}_2$ (see fig. 4). Let g_0 be the bisector of the line $\overline{\boldsymbol{s}_1\boldsymbol{s}_2}$. By construction, $\boldsymbol{c}_0$ is on g_0. Define $\boldsymbol{c}_1$ and $\boldsymbol{c}_2$ as the points on g_0 whose distance from $\boldsymbol{s}_1$ and $\boldsymbol{s}_2$ is r, and draw the circles b_1 and b_2 with radius r around them. Now, the boundary of A cannot lie inside either $b_1 \setminus b_2$ or $b_2 \setminus b_1$, because otherwise A could not be r-regular. The areas where ∂A may run are marked dark gray in fig. 4. Since $p < r$, there can be no further intersections between $\partial K_p(\boldsymbol{c}_0)$ and ∂A besides $\boldsymbol{s}_1$ and $\boldsymbol{s}_2$.

On g_0, mark the points $\boldsymbol{c}_3$ between $\boldsymbol{c}_0$ and $\boldsymbol{c}_1$, and $\boldsymbol{c}_3'$ between $\boldsymbol{c}_0$ and $\boldsymbol{c}_2$, such that $|\overline{\boldsymbol{c}_1\boldsymbol{c}_3}| = |\overline{\boldsymbol{c}_2\boldsymbol{c}_3'}|$ and $\min(|\overline{\boldsymbol{c}_0\boldsymbol{c}_3}|, |\overline{\boldsymbol{c}_0\boldsymbol{c}_3'}|) = r' = r - p$. Due to the triangle inequality, and since $p < r$, such a configuration always exists. We prove the lemma for the circle b_{out} around $\boldsymbol{c}_3$, b_{in} around $\boldsymbol{c}_3'$ is treated analogously.

Let $b_3 = b_{\text{out}}$ be the circle around $\boldsymbol{c}_3$ with radius r', and b_3' the circle around $\boldsymbol{c}_3$ that touches $\boldsymbol{s}_1$ and $\boldsymbol{s}_2$ (fig. 5 left). Consider a point $\boldsymbol{c}_4$ on ∂b_3 and draw the circle b_4 with radius p around $\boldsymbol{c}_4$. This circle corresponds to the footprint of the PSF centered at $\boldsymbol{c}_4$. Now we would like to compare the result of the convolution $k_p \star \chi_A$ at $\boldsymbol{c}_0$ and $\boldsymbol{c}_4$. The convolution results are determined by the amount of overlap between A and $b_0 = K_p(\boldsymbol{c}_0)$ and $b_4 = K_p(\boldsymbol{c}_4)$ respectively. To compare $b_0 \cap A$ and $b_4 \cap A$, we split the two circles into subsets according to fig. 5 center (only b_0, b_4 and b_3' are shown in this figure). Circle b_0 consists of the subsets f_1, f_2, f_3, f_4, whereas b_4 consists of f_1, f_2, f_3', f_4'. The subsets f_1 and f_2 are shared by both circles, while due to symmetry f_3, f_3' and f_4, f_4' are mirror images of each other. In terms of the subsets, we can express the convolution results as follows:

$$(k_p \star \chi_A)(\boldsymbol{c}_0) = \frac{\|f_1 \cap A\| + \|f_2 \cap A\| + \|f_3 \cap A\| + \|f_4 \cap A\|}{\|K_p\|}$$

$$(k_p \star \chi_A)(\boldsymbol{c}_4) = \frac{\|f_1 \cap A\| + \|f_2 \cap A\| + \|f_3' \cap A\| + \|f_4' \cap A\|}{\|K_p\|}$$

By straightforward algebraic manipulation we get:

$$\|K_p\| \left((k_p \star \chi_A)(\boldsymbol{c}_0) - (k_p \star \chi_A)(\boldsymbol{c}_4)\right) = \\ \|f_3 \cap A\| - \|f_3' \cap A\| + \|f_4 \cap A\| - \|f_4' \cap A\| \quad (1)$$

Since the radius of b_3' is smaller than r, and its center $\boldsymbol{c}_3$ is between $\boldsymbol{c}_0$ and $\boldsymbol{c}_1$, the boundary $\partial b_3'$ intersects ∂A only at s_1 and s_2. It follows that subset f_3 is completely inside of A, whereas f_4' is completely outside of A. Hence, we have $\|f_3 \cap A\| = \|f_3\| = \|f_3'\|$ and $\|f_4' \cap A\| = 0$. Inserting this into (1), we get

$$\|K_p\| \left((k_p \star \chi_A)(\boldsymbol{c}_0) - (k_p \star \chi_A)(\boldsymbol{c}_4)\right) = \|f_3'\| - \|f_3' \cap A\| + \|f_4 \cap A\| > 0 \quad (2)$$

Thus, the gray level at $\boldsymbol{c}_4$ is smaller than l. When $\boldsymbol{c}_4$ is moved further away from $\boldsymbol{c}_0$, the subset f_2 will eventually disappear from the configuration (fig. 5 right). If $\boldsymbol{c}_3$ is outside of b_0, f_1 will finally disappear as well. It can easily be checked that (2) remains valid in either case. Due to the definition of $\boldsymbol{c}_3$, no other configurations are possible. Therefore, the gray values on the boundary ∂b_{out} are below l everywhere except at $\boldsymbol{c}_0$.

It remains to prove the same for the interior of b_{out}. Suppose the gray level at point $\boldsymbol{c} \in b_{\text{out}}^0$ were $l' \geq l$. By what we have already shown, the associated level line $\partial L_l'$ cannot cross the boundary ∂b_{out} (except at the single point c_0 if $l' = l$). So it must form a closed curve within b_{out}. However, this curve would cross some normal of ∂A twice, in contradiction to lemma 5. This implies the claim for outside circles. The proof for inside circles proceeds analogously. □

We conclude that the shape of the level sets L_l is quite restricted:

Theorem 2. *Let A be an r-regular set, and L_l any level set of $k_p \star \chi_A$, where k_p is a flat disk-like point spread function with radius $p < r$. Then L_l is r'-regular (with $r' = r - p$) and p-similar to A.*

Proof. The proof of r'-regularity follows directly from the definition of r-regularity and lemma 6.

Now assume that there exists a homeomorphism $f : \mathbb{R}^2 \to \mathbb{R}^2$ such that $f(A) = L_l$ and $\forall \boldsymbol{x} \in \mathbb{R}^2 : |f(\boldsymbol{x}) - \boldsymbol{x}| \leq p$. This homeomorphism would induce a homeomorphism from A to L_l. Due to the embedding of f in $\mathbb{R}^2$, the homotopy trees of A and $f(A)$ would be equal. Since $|f(\boldsymbol{x}) - \boldsymbol{x}| \leq p$, the Hausdorff distance between ∂A and $f(\partial A)$ would be at most p. Thus, the existence of such a homeomorphism is sufficient to prove p-similarity.

The required homeomorphism can indeed be constructed: Because of the restricted curvature of ∂A, the normals of ∂A cannot intersect within the p-strip A_p around ∂A (cf. [1,2]). Therefore, due to 5, every point $\boldsymbol{s}$ on ∂A can be translated along its normal towards a unique point on the given level line ∂L_l and vice versa. The distance between s and its image is $\leq p$. This mapping can be extended to the entire $\mathbb{R}^2$-plane in the usual way, so that we get a homeomorphism with the desired properties. □

This finally allows us to show what happens during the digitization of a set A that was subjected to blurring with a PSF:

Theorem 3 (sampling theorem for blurred binary images). *Let A be an r-regular set, L_l any level set of $k_p \star \chi_A$, where k_p is a flat disk-like point spread function with radius $p < r$, and S a grid with maximum pixel radius $r'' < r - p$. The S-reconstruction $\hat{L}_l$ of L_l is $(p + r'')$-similar to A.*

Proof. By theorem 2, L_l is r'-regular and p-topologically similar to A. By theorem 1, the S-reconstruction of an r'-regular set with an r''-grid ($r'' < r'$) is r''-similar to the original set. Thus A, L_l and $\hat{L}_l$ are topologically equivalent and have the same homotopy tree. Due to the triangle inequality of the Hausdorff metric, the Hausdorff distance between A and $\hat{L}_l$ is at most $p + r''$. □

Corollary 2. *Since $r'' + p < r$, any S-reconstruction of L_l is r-topologically similar to A, regardless of how the grid is rotated and translated relative to A.*

5 Conclusions

Our results are intuitively very appealing: When we digitize an ideal binary image with *any* r''-grid, we can properly reconstruct a shape if it is r'-regular with $r' > r''$. But when the image is first subjected to blurring with a PSF of radius p, the set must be r-regular with $r > r'' + p$. In other words, the radius of the PSF must be added to the radius of the grid pixels to determine the regularity requirements for the original shape. It should also be noted that $r > r'' + p$ is a tight bound, which for instance would be reached if A consisted of a circle of radius r, and the threshold was 1 – in this case, any smaller circle could get lost in the reconstruction. However, for a single, pre-selected threshold a better bound can be derived.

Our result is closely related to the findings of Latecki et al. [1,2] about v-digitization (and thus also square subset digitization and intersection digitization). In their approach, the grid must be square with sampling distance h, and the PSF is an axis aligned flat square with the same size as the pixels. Then, the pixel and PSF radius are both $r'' = p = h/\sqrt{2}$, and the original shape must be r-regular with $r > r'' + p = \sqrt{2}h$. This is exactly the same formula as in our case. We conjecture that our results can be generalized to a much wider class of radially symmetric PSFs, but we can't prove this yet.

References

1. Latecki, L.J., Conrad, C., Gross, A.: *Preserving Topology by a Digitization Process.* Journal of Mathematical Imaging and Vision **8**, 131–159, 1998.
2. Latecki, L.J.: *Discrete Representation of Spatial Objects in Computer Vision.* Kluwer Academic Publishers, Dordrecht, 1998.
3. Pavlidis, T.: *Algorithms for Graphics and Image Processing.* Computer Science Press, Rockville, Maryland. 1982.
4. Ronse, C., Tajine, M.: *Discretization in Hausdorff Space.* Journal of Mathematical Imaging and Vision **12**, 219–242, 2000.
5. Serra, J.: *Image Analysis and Mathematical Morphology* Academic Press, New York, 1982.