A Simple Algorithm for Homeomorphic Surface Reconstruction

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Abstract

The problem of computing a piecewise linear approximation to a surface from a set of sample points on the surface has been a focus of research in solid modeling and graphics due to its many applications. The input to this *surface reconstruction* problem consists of the three dimensional coordinates of the sampled points. The crust algorithm of [1] reconstructs a surface with topological and geometric guarantees using the Voronoi diagram of the input point set. We present new observations that simplify both the algorithm and the proofs for the crust, and we give for the first time a proof that the crust is homeomorphic to the input surface.

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1 Introduction

A number of applications ranging over CAD, computer graphics and mathematical modeling involve the computation of a piecewise linear approximation to a surface from sample points. In practice, the input point set is generated using a variety of tools such as laser range scanners, contact probe digitizers and medical imageries like CT or MRI scanners. The input may contain additional information such as estimated surface normals, which can be quite useful; for example, see [7]. Although such additional information can be helpful, it is not always available. Hoppe et al. [15] pointed out the need for addressing the problem of surface reconstruction under a general setting, where no additional information is available other than the space co-ordinates of the sample points. The two dimensional version of the problem, namely the curve reconstruction in plane from sample co-ordinates has been well researched. A variety of approaches [2, 4, 5, 8, 9, 13, 14, 17] are known to work with theoretical guarantees.

In three dimensions, only a few algorithms known to date provide performance guarantees. Hoppe et. al [15] presented an algorithm based on *zero sets* of a signed distance function. Using the same basic idea, Curless and Levoy gave an algorithm which reconstructs surfaces very rapidly from large, noisy laser range scanner data sets [7]. Recently, Boissonnat and Cazals used the zero set of an implicit function defined with natural interpolation to reconstruct smooth surfaces [6]. Edelsbrunner reports success with a proprietary commercial program [10]. The α-shapes algorithm, as described by [11] by Edelsbrunner and Mücke is useful for reconstructing surfaces that have been sampled with uniform density. However, these results do not provide any assurance that the output surface is topologically equivalent to the sampled surface. Clearly, it is not possible to compute a surface that is faithful to the topology and geometry of the original unless the sampling is sufficiently dense.

Amenta and Bern [1] proposed that the sampling density be proportional to the *local feature size* of the surface and presented an algorithm based on Delaunay triangulations. They proved that the output of their algorithm, the *crust* of the sample set, is geometrically close to the surface S assuming that S is a smooth 2-manifold without boundary and that the sampling is sufficiently dense. Their algorithm uses two passes of Voronoi diagram (or its dual Delaunay triangulation) computation, and also two postprocessing steps, called *filtration by normals* and *trimming*. In this paper, we propose a simpler, single pass Delaunay algorithm for reconstruction. Also, Amenta and Bern [1] did not prove that the crust is homeomorphic to S. In this paper, we present the first such proof.

Our algorithm is based on the following observation. Let T be a set of triangles spanning all sample points and satisfying three conditions, namely, I. T contains all triangles whose dual Voronoi edges intersect S, II. each triangle in T is small, that is, their circumcircle has a small radius compared to the local feature size, III. all triangles in T are "flat", that is, their normals make small angles with the normals to the surface at their vertices. A piecewise linear 2-manifold spanning all samples can be extracted from T due to condition I. Then, using conditions II and III we show that any piecewise linear 2-manifold extracted from T that spans all of its vertices must be homeomorphic to S.

The algorithm computes T from the Delaunay triangulation as follows. Let p be a sample point and e be a Voronoi edge in the Voronoi cell of p. The algorithm determines if e has a point x, where px makes an angle close to $\pi/2$ with an estimated normal at the sample point p. If this condition is satisfied for all three Voronoi cells adjacent to e, its dual is included in the candidate set T. We prove that T satisfies conditions I and II and condition II implies condition III. Thus, one needs not check the condition III explicitly.

Our observations not only simplifies the crust algorithm of [1] by eliminating one Delaunay pass and normal trimming step, but also simplifys the proofs.

The paper is organized as follows. Section 2 contains definitions and the preliminaries. Section 3 describes the algorithm. Section 4 contains the proofs of all three conditions I, II and III. Section 5 shows the homeomorphism between the output and the sampled surface. We describe our implementation experience in Section 6 and conclude in Section 7.

2 Definitions and Preliminaries

We assume that the sampled surface S is a smooth manifold without boundary, embedded in $\mathbf{R}^3$. We adopt the following definition of sampling density from [1].

2.1 Medial axis and ϵ-sampling

The *medial axis* of a surface S in $\mathbf{R}^3$ is the closure of points that have more than one closest point on S. The *local feature size*, $f(p)$, at a point p on S is the least distance of p to the medial axis. The *medial balls* at p are defined as the balls that touch S tangentially at p and have center on the medial axis. Notice that $f(p)$ is not necessarily same as the radius of the *medial balls* at p. A very useful property of $f(\cdot)$ is that it is 1-Lipschitz, that is, $f(p) \leq f(q) + |pq|$ for any two points p, q on S. A point set P is called an ϵ-sample of a surface S if every point $p \in S$ has a sample within a distance of $\epsilon f(p)$.

2.2 Restricted Delaunay triangulation

We assume that the input sample $P \in \mathbf{R}^3$ is in general position to keep the description simple. Let D_P and V_P denote the Delaunay triangulation and the Voronoi diagram of P. A Voronoi cell $V_p \in V_P$ for each point $p \in P$ is defined as the set of points $x \in \mathbf{R}^3$ such that $|px| \leq |qx|$ for any $q \in P$ and $q \neq p$. The Delaunay triangulation has an edge pq if and only if V_p, V_q share a face, has a triangle pqr if and only if V_p, V_q and V_r share an edge, and a tetrahedron $pqrs$ if and only if V_p, V_q, V_r and V_s share a Voronoi vertex.

Consider the restriction of V_P on the surface S which defines the *restricted Voronoi diagram* $V_{P,S}$ containing *restricted Voronoi cells* $V_{p,S} = V_p \cap S$. The dual of these restricted Voronoi cells defines the *restricted Delaunay triangulation* $D_{P,S}$. Specifically, an edge pq is in $D_{P,S}$ if and only if $V_{p,S} \cap V_{q,S}$ is nonempty; a triangle pqr is in $D_{P,S}$ if and only if $V_{p,S} \cap V_{q,S} \cap V_{r,S}$ is nonempty. Assuming that S does not pass through a Voronoi vertex in general position we do not have any tetrahedron in $D_{P,S}$. A result of Edelsbrunner and Shah [12] shows that the underlying space of $D_{P,S}$ is homeomorphic to S if the following *closed ball property* holds. Each $V_{p,S}$ is a topological 2-ball, each nonempty pairwise intersection $V_{p,S} \cap V_{q,S}$ is a topological 1-ball, and each nonempty triplet-wise intersection $V_{p,S} \cap V_{q,S} \cap V_{r,S}$ is a single point, that is, a 0-ball. Amenta and Bern used this result to show that if P is a ϵ-sample of S with $\epsilon \leq 0.1$, then $V_{P,S}$ satisfies the closed ball property and hence D_P contains the set of triangles of $D_{P,S}$ whose underlying space is homeomorphic to S.

2.3 Conditions for homeomorphism

Our algorithm extracts a piecewise linear manifold N from a set of *candidate triangles* T that satisfy the following three conditions.

I. Restricted Delaunay condition. The set of triangles include all restricted Delaunay triangles.

II. Small triangle condition. The circumcircle of each triangle $t \in T$ is small, that is, its radius is $O(\epsilon)f(p)$, where p is a vertex of t.

III. Flat triangle condition. The normal to each $t \in T$ makes small angle $O(\epsilon)$ with the surface normal at the vertex p, where p contains the largest face angle inside t.

The algorithm makes sure that the candidate triangles computed by it satisfy conditions I and II. We show that small triangles are always flat, i.e., condition II implies condition III. Condition I is necessary for the manifold extraction step of the algorithm and conditions II and III are used to show that any piecewise linear 2-manifold extracted from T which spans all of its vertices is homeomorphic to S.

3 Algorithm

The algorithm selects the candidate triangles using *co-cones* at each sample point and then extracts a manifold from them using a step similar to [1].

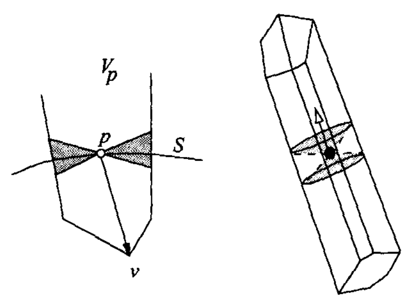

Figure 1: The co-cone for a sample in two dimensions (left), and three dimensions (right). In the left the co-cone is shaded, in the right its boundary is shaded.

Co-cones.

The normal to S at each sample point is estimated using the concept of "pole" as introduced in [1]. For each Voronoi cell V_p, the Voronoi vertex that is farthest from the sample point p is taken as the pole. The line through p and the pole is almost normal to S and is called the *estimated normal line* at p; see Figure 1. For an angle θ, we define the complemented cone called *co-cone* at a sample p as the complement of the double cone with apex p which makes an angle $\pi/2-\theta$ with the estimated normal line at p. We determine the set of Voronoi edges in V_p that intersect the co-cone of p. The dual triangles of these Voronoi edges for each p constitute the candidate set T. We will argue that these set of triangles satisfy the three conditions I, II and III if θ is sufficiently small, say $\theta = \pi/8$.

The set of Voronoi edges intersecting co-cones are computed as follows. Let n_p denote the line normal to S at p and $v \in V_p$ be the farthest Voronoi vertex from p in V_p. Denote any ray from p to a point $y \in V_p$ as $\vec{y}$. For each $p \in P$, the algorithm determines the set of Voronoi edges in V_p that has a point x where $\vec{x}$ makes an angle in the range $I = [\pi/2-\theta, \pi/2+\theta]$ with $\vec{v}$, where θ is chosen to be $\pi/8$. Let e be an edge in the Voronoi cell V_p, and w_1, w_2 be its two endpoints. We compute $\angle \vec{w_1}\vec{v}$ and $\angle \vec{w_2}\vec{v}$ and check if the range of angles determined by these two angles intersects the desired range I. If it does, we mark e. We include e in E if it is marked for all three Voronoi cells adjacent with e.

Manifold extraction.

The candidate set T is the dual Delaunay triangles of E. A piecewise linear manifold N is extracted from T using the last step of the crust algorithm [1]. This step first deletes all the triangles incident on *sharp edges*. An edge is called *sharp* if the angle between the two consecutive triangles around the edge is more than $3\pi/2$. An edge with a single incident triangle is also sharp. Deleting such triangles are safe since the restricted Delaunay triangles are not incident with sharp edges and they are included in the set (condition I). Next, a depth-first walk over the adjacency graph of the remaining triangles extracts the outer boundary N of the underlying space of them. The result in section 5 implies that this boundary is homeomorphic to S.

Theorem 1 *The above algorithm computes a piecewise linear surface homeomorphic to a surface for which P is an ϵ-sample with $\epsilon \leq 0.06$.*

We remark that the condition $\epsilon \leq 0.06$ comes from the fact that our proof of homeomorphism in section 5 requires this range of ϵ. Actually the three conditions are satisfied with larger value of ϵ. The required value of ϵ deteriorates in Lemma 13 that is used in the homeomorphism proof.

4 Conditions

We need the following two lemmas from [1] that says that the pole estimates the normal at the sample point.

Lemma 2 *Let y be any point in V_p so that $|py| \geq \delta f(p)$ for $\delta > 0$. The acute angle between $\vec{y}$ and n_p is less than $\sin^{-1}\frac{\epsilon}{\delta(1-\epsilon)} + \sin^{-1}\frac{\epsilon}{1-\epsilon}$.*

Using Lemma 2 and the fact that $|pv| > f(p)$, it is shown in [1] that $\vec{v}$ approximates the normal direction n_p quite well.

Lemma 3 *The acute angle between n_p and $\vec{v}$ is less than $2\sin^{-1}\frac{\epsilon}{1-\epsilon}$.*

We also need the following lemma from [1].

Lemma 4 *Let p, q be two points on S so that $|pq| < \rho \min\{f(p), f(q)\}$ with $\rho < 1/3$. Then the angle between n_p and n_q is at most $\frac{\rho}{1-3\rho}$.*

4.1 Restricted Delaunay condition

Condition I requires that the restricted Delaunay triangles are in T. First, we need a technical lemma about local flatness of S.

Lemma 5 *Let y be any point in $V_{p,S}$. The acute angle between n_p and $\vec{y}$ is larger than $\pi/2 - \epsilon$.*

Proof. The distance $|yp| \leq \epsilon f(y)$, since $y \in V_{p,S}$, and by the Lipschitz condition $f(y) \leq f(p) + |py|$ giving $f(y) \leq \frac{f(p)}{1-\epsilon}$, and hence $|py| \leq \epsilon f(y) \leq \frac{\epsilon}{1-\epsilon} f(p)$. Point y must be outside of both medial balls at p, each of which has radius at least $f(p)$. The angle in question is thus at least $\pi/2 - \sin^{-1} \frac{\epsilon}{2(1-\epsilon)}$, which is at least $\pi/2 - \epsilon$ for $\epsilon \leq 0.1$. ◻

Theorem 6 *For $\epsilon \leq 0.1$, all restricted Delaunay triangles are in T.*

Proof. It is shown in [1] that the restricted Delaunay triangulation is homeomorphic to S for $\epsilon \leq 0.1$. Let e be the dual edge of a restricted Delaunay triangle. Consider the point $y = e \cap S$. We have $y \in V_{p,S}$ for some $p \in P$. We show that $\vec{y}$ makes an angle in the range I with the pole direction $\vec{v}$. The result follows from our choice of E. The acute angle between n_p and $\vec{y}$ is larger than $\pi/2 - \epsilon$ by Lemma 5. Therefore $\angle \vec{y}\vec{v} \in [\pi/2 - \epsilon - \alpha, \pi/2 + \epsilon + \alpha]$, where α is the acute angle between $\vec{v}$ and n_p. Plugging in the upper bound of α from Lemma 3 we obtain that $\angle \vec{y}\vec{v} \in I$ when $\epsilon \leq 0.1$. ◻

4.2 Small triangle condition

Interpreting Lemma 2 differently and plugging in a value of $\delta = \frac{1.3\epsilon}{1-\epsilon}$ we obtain the following.

Corollary 7 *Let x be any point in V_p so that the acute angle between $\vec{x}$ and n_p is at least $\pi/2 - \theta - 2\sin^{-1} \frac{\epsilon}{1-\epsilon}$. Then $|px| < \frac{1.3\epsilon}{1-\epsilon} f(p)$ for $\theta = \pi/8$ and $\epsilon \leq 0.09$.*

Proof. If the acute angle between $\vec{x}$ and n_p is at least $\alpha = \sin^{-1} \frac{\epsilon}{\delta(1-\epsilon)} + \sin^{-1} \frac{\epsilon}{1-\epsilon}$, then $|px| < \delta f(p)$ according to Lemma 2. With $\delta = \frac{1.3\epsilon}{1-\epsilon}$ we have

$$\alpha = \sin^{-1} \frac{1}{1.3} + \sin^{-1} \frac{\epsilon}{1-\epsilon}$$

which is less than $\pi/2 - \theta - 2\sin^{-1} \frac{\epsilon}{1-\epsilon}$ for $\theta = \pi/8$ and $\epsilon \leq 0.09$. ◻

Theorem 8 *Let r denote the radius of any triangle $t \in T$. Then, for each vertex p of t, $r \leq \frac{1.3\epsilon}{1-\epsilon} f(p)$ for $\epsilon \leq 0.09$.*

Proof. Let e be the dual edge of t. By our choice of e, there is a point $x \in e$ so that $\vec{x}$ makes an angle in the range $I = [\pi/2 - \theta, \pi/2 + \theta]$ with $\vec{v}$ for $\theta = \pi/8$. Taking into account the angle between $\vec{v}$ and n_p we conclude that this ray makes an acute angle more than $\pi/2 - \theta - 2\sin^{-1} \frac{\epsilon}{1-\epsilon}$ with n_p. From Corollary 7, $|px| < \frac{1.3\epsilon}{1-\epsilon} f(p)$. The radius of the circumcircle of t cannot be more than this. ◻

4.3 Flat triangle condition

Theorem 9 *Any triangle $t \in T$ has a normal that makes an acute angle no more than $\alpha + \sin^{-1}(\frac{2}{\sqrt{3}} \sin 2\alpha)$ with n_p where p is the vertex of t with the largest face angle, $\alpha \leq sin^{-1} \frac{1.3\epsilon}{1-\epsilon}$, and $\epsilon \leq 0.09$.*

Proof. Consider the medial balls M_1 and M_2 touching S at p with the centers on the medial axis. Let D be the ball with t lying on a diametric plane; refer to Figure 2. The radius r of D is equal to the radius of the circumcircle of t. Denote the circles of intersection of D with M_1 and M_2 as C_1 and C_2 respectively. The normal to S at p passes through m, the center of M_1. This normal makes an angle less than α with the normals to the planes of C_1 and C_2, where

$$\begin{aligned} \alpha &\leq \sin^{-1} r/|pm| \\ &\leq \sin^{-1} \frac{1.3\epsilon}{1-\epsilon} \end{aligned}$$

since $|pm| \geq f(p)$ by definition and $r \leq \frac{1.3\epsilon}{1-\epsilon} f(p)$ by Theorem 8. This angle bound also applies to the plane of C_2, which implies that the planes of C_1 and C_2 make a wedge, say W, with an acute dihedral angle no more than 2α.

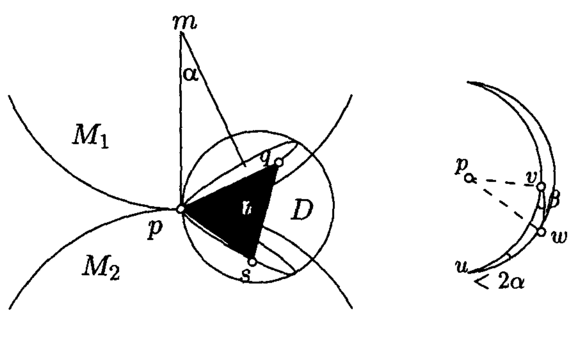

Figure 2: Normal to a small triangle and the normal to S at the vertex with the largest face angle.

The vertices q, s of t cannot lie inside M_1 or M_2. This implies that t lies completely in the wedge W. Consider a cone at p inside the wedge W formed by the three planes; π_t, the plane of t, π_1, the plane of C_1 and π_2, the plane of C_2. A unit sphere centered around p intersects the cone in a spherical triangle uvw, where u, v and w are the points of intersections of the lines $\pi_1 \cap \pi_2$, $\pi_t \cap \pi_1$ and $\pi_t \cap \pi_2$ respectively with the unit sphere. See the picture on right in Figure 2. Without the loss of generality, assume that the angle $\angle uvw \leq \angle uwv$. We have the following facts. The arc length of wv, denoted $|wv|$, is at least $\pi/3$ since p subtends the largest angle in t and t lies completely in the wedge W. The spherical angle $\angle vuw$ is less than or equal to 2α. We are interested in the spherical angle $\beta = \angle uvw$ which is also the acute dihedral angle between the planes of t and C_1. By standard *sine laws* in spherical geometry, we have $\sin\beta = \sin|uw|\frac{\sin\angle vuw}{\sin|wv|} \leq \sin|uw|\frac{\sin 2\alpha}{\sin|wv|}$. If $\pi/3 \leq |wv| \leq 2\pi/3$, we have $\beta \leq \sin^{-1}\frac{2}{\sqrt{3}}\sin 2\alpha$. For the range $2\pi/3 < |wv| < \pi$, we use the fact that $|uw| + |wv| \leq \pi$ since $\angle vuw \leq 2\alpha < \pi/2$ for sufficiently small ϵ. So, in this case $\frac{\sin|uw|}{\sin|wv|} < 1$. Thus, $\beta \leq \sin^{-1}\frac{2}{\sqrt{3}}\sin 2\alpha$.

The normals to t and S at p make an acute angle at most $\alpha + \beta$ proving the theorem. ☐

5 Homeomorphism

In this section, we will show a homeomorphism between S and any piecewise-linear surface N made up of candidate triangles and spanning all sample points completing the proof of Theorem 1. We define the homeomorphism explicitly, using a function. We initially define a map μ on all of $\mathbf{R}^3$, and then use its restriction to N. Let $\mu : \mathbf{R}^3 \to S$ map each point $q \in \mathbf{R}^3$ to the closest point of S.

Lemma 10 *The restriction of μ to N is a well defined and continuous function $\mu : N \to S$.*

Proof. The discontinuities of μ as a map on $\mathbf{R}^3$ are exactly the points of the medial axis. If some point q had more than one closest point on the surface, q would be a point of the medial axis; but it follows from Theorem 8 that every point $q \in N$ is within $\frac{1.3\epsilon}{1-\epsilon} f(p)$ of a triangle vertex $p \in S$. Similarly, μ is continuous except at the medial axis of S, so that since N is continuous and avoids the medial axis, μ is continuous on N. ☐

The function μ defines a homeomorphism between N and S if it is continuous, one-to-one and onto. Our approach will be first to show that μ is well-behaved on the samples themselves, and then show that this behavior continues in the interior of each triangle of N.

Lemma 11 *Let p be a sample and let m be the center of a medial ball M tangent to the surface at p. No candidate triangle intersects the interior of the segment pm.*

Proof. In order to intersect segment pm, a candidate triangle t would have to intersect M, and

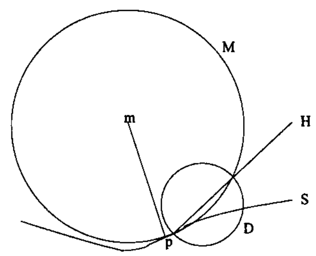

Figure 3: Proof of Lemma 11.

so would the smallest Delaunay ball D of t. Let H be the plane of the circle where the boundaries of M and D intersect. We show that H separates the interior of pm and t.

The plane H decomposes the ball M into two caps $M^+ = M \cap H^+$, $M^- = M \cap H^-$ and the disk $M \cap H$, where H^+ and H^- are the two open half spaces delimited by H. Similarly, D is also decomposed by H into D^+, D^- and $D \cap H$. It follows from sphere geometry that if $M^+ \subset D^+$ then D^+ and M^- lie on opposite sides of H. Without loss of generality assume $M^+ \subset D^+$.

Since the vertices of t lie on S and hence not in the interior of M, $t \subset D^+$. Since D is Delaunay, p cannot lie in the interior of D hence $p \notin M^+$. We claim that $m \notin M^+$ either. (see Figure 3.) Indeed if m were inside $M^+ \subset D^+$, the radius of D would be at least $1/2\ f(p')$ for any vertex p' of t, contradicting, by Theorem 8, the assertion that t is a candidate triangle. Therefore p, m and hence the segment pm lies in M^- proving that H separates t and pm. ◻

Since any point q such that $\mu(q) = p$ lies on such an open segment pm, we have the following.

Corollary 12 *The function μ is one-to-one from N to every sample p.*

In what follows, we will show that μ is indeed one-to-one on all of N. One more geometric preliminary is required. We already know that the normal of a candidate triangle t is close to the surface normal at its vertex with the largest angle (Theorem 9). To complete the proof of homeomorphism, we need to show a stronger assertion: that the triangle normal agrees with the surface normal at $\mu(q)$ for every $q \in t$.

Lemma 13 *Let q be a point on triangle $t \in N$. The acute angle between the surface normal n_q at $\mu(q)$ and the normal to t measures at most $59°$ degrees for $\epsilon \leq 0.06$.*

Proof. The circumcircle of t is small; the distance from q to the vertex p of t with the largest angle is $2\delta\ f(p)$, with $\delta = \frac{1.3\epsilon}{1-\epsilon}$, by Theorem 8. The point $\mu(q) = x$ is the closest point on S to q. Since there is a sample, namely, a vertex of t within $\delta f(p)$ away from q, we have $|qx| \leq \delta f(p)$. We are interested in an upper bound on $|px|$ so that Lemma 4 can be applied.

It can be shown that $|px|$ is maximized when the angle pqx is a right angle. Thus, $|px| \leq \sqrt{5}\delta f(p) \leq 0.19 f(p)$ for $\epsilon \leq 0.06$. Also, $f(p) \leq 1.23 f(x)$ by Lipschitz property of $f(\cdot)$ giving $|px| \leq 0.23 f(x)$. Thus, applying Lemma 4 the angle between n_x and n_p is less than $43°$ taking $\rho = 0.23$. The angle between the triangle normal of t and n_p is less than $16°$ for $\epsilon \leq 0.06$ (Theorem 9). Thus, the triangle normal and n_x makes an angle less than $59°$. ◻

Now, we show that selecting a piecewise-linear manifold N from the set of candidate triangles would be sufficient for reconstruction, under an additional mild assumption. A pair of triangles $t_1, t_2 \in N$ are *adjacent* if they share at least one common vertex.

Assumption: Two adjacent triangles meet at their common vertex at an angle of greater than $\pi/2$.

This assumption excludes manifolds which contain sharp folds and, for instance, flat tunnels.

Our proof proceeds in three short steps. We show that μ induces a homeomorphism on each triangle, then on each pair of adjacent triangles, and finally on N as a whole.

Lemma 14 *Let U be a region contained within one triangle $t \in N$ or in adjacent triangles of*

N. The function μ defines a homeomorphism between U and $\mu(U) \subset S$.

Proof. We know that μ is well-defined and continuous on U, so it only remains to show that it is one-to-one. First, we prove that if U is in one triangle t, μ is one-to-one. For a point $q \in t$, the vector $\vec{n_q}$ from $\mu(q)$ to q is perpendicular to the surface at $\mu(q)$; since S is smooth the direction of $\vec{n_q}$ is unique and well defined. If there was some $y \in t$ with $\mu(y) = \mu(q)$, then q, $\mu(q)$ and y would all be colinear and t itself would have to contain the line segment between q and y, contradicting Lemma 13, which says that the normal of t is nearly parallel to $\vec{n_q}$.

Now, we consider the case in which U is contained in more than one triangle. Let q and y be any two points in U such that $\mu(q) = \mu(y)$, and let v be a common vertex of the triangles that contain U. Since μ is one-to-one in one triangle, $\mu(q) = \mu(y)$ implies that q and y lie in the two distinct triangles t_q and t_y. Let $\vec{n}$ be the surface normal at $\mu(q) = \mu(y)$. Since the line supporting $\vec{n}$ passes through both t_q and t_y, and the acute angles between the triangle normals of t_q, t_y and $\vec{n}$ are at most 59° (Lemma 13), t_q and t_y must meet at v at an acute angle. This would contradict the Assumption, which is that t_q and t_y meet at v at an obtuse angle. Hence there are no two points in y, q with $\mu(q) = \mu(y)$. □

We finish the theorem using a theorem from topology.

Theorem 15 *The mapping μ defines a homeomorphism from the triangulation N to the surface S for $\epsilon \leq 0.06$.*

Proof. Let $S' \subset S$ be $\mu(N)$. We first show that (N, μ) is a *covering space* of S'. Informally, (N, μ) is a covering space for S' if function μ maps N smoothly onto S', with no folds or other singularities; see Massey [16], Chapter 5. Showing that (N, μ) is a covering space is weaker than showing that μ defines a homeomorphism, since, for instance, it does not preclude several connected components of N mapping onto the same component of S', or more interesting behavior, such as a torus wrapping twice around another torus to form a *double covering.*

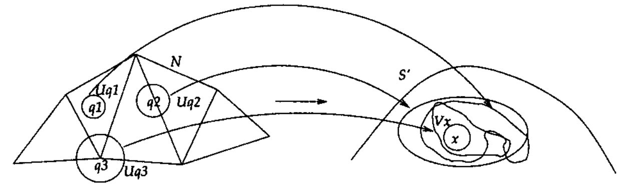

Figure 4: Proof of Theorem 15.

Formally, the (N, μ) is a covering space of S' if, for every $x \in S'$, there is a path-connected *elementary neighborhood* V_x around x such that each path-connected component of $\mu^{-1}(V_x)$ is mapped homeomorphically onto V_x by μ.

To construct such an elementary neighborhood, note that the set of points $|\mu^{-1}(x)|$ corresponding to a point $x \in S'$ is non-zero and finite, since μ is one-to-one on each triangle of N and there are only a finite number of triangles. For each point $q \in \mu^{-1}(x)$, we choose an open neighborhood U_q of around q, homeomorphic to a disk and small enough so that U_q is contained only in triangles that contain q.

We claim that μ maps each U_q homeomorphically onto $\mu(U_q)$. This is because it is continuous, it is onto $\mu(U_q)$ by definition, and, since any two points x and y in U_q are in adjacent triangles, it is one-to-one by Lemma 14.

Let $U'(x) = \cap_{q \in \mu^{-1}(x)} \mu(U_q)$, the intersection of the maps of each of the U_q. $U'(x)$ is the intersection of a finite number of open neighborhoods, each containing x, so we can find an open disk V_x around x. V_x is path connected, and each component of $\mu^{-1}(V_x)$ is a subset of some U_q and hence is mapped homeomorphically onto V_x by μ. Thus (N, μ) is a covering space for S'.

We now show that μ defines a homeomorphism between N and S'. Since N is onto S' by definition, we need only show that μ is one-to-one. Consider one connected component G of S'. A theorem of algebraic topology (see eg.

Massey [16], Chapter 5 Lemma 3.4) says that when (N, μ) is a covering space of S', the sets $\mu^{-1}(x)$ for all $x \in G$ have the same cardinality. We now use Corollary 12, that μ is one-to-one at every sample. Since each connected component of S contains some samples, it must be the case that μ is everywhere one-to-one, and N and S' are homeomorphic.

Finally, we show that $S' = S$. We must have S' to be closed and compact since N is. So S' cannot include part of a connected component of S, and hence S' must consist of a subset of the connected components of S. Since every connected component of S contains a sample s (actually many samples), and $\mu(s) = s$, all components of S belong to S', $S' = S$, and N and S are homeomorphic. ◻

6 Implementation

We have implemented the algorithm. Two outputs are shown in Figure 5. The algorithm was quite easy to implement with the well known *qhull* code for Delaunay triangulation. In the figure we show the output of co-cone step. The surface is computed correctly almost everywhere except at the boundaries and near sharp features. This is expected since the algorithm is not geared to handle boundaries or sharp features. The program ran on a SUN machine with the 300Mhz processor and 256 MB memory for less than a couple of minutes on all tested examples. For example, the foot took 153 seconds. It is reported in [3] that the same data took 15 minutes on a SGI Onyx machine with 512 MB memory. The difference can be explained by two factors; first, this algorithm requires only one Delaunay triangulation step, and second, the implementation of [3] used the exact-arithmetic Delaunay triangulation program `hull`, which we have observed to be about four times slower than `qhull` on these inputs.

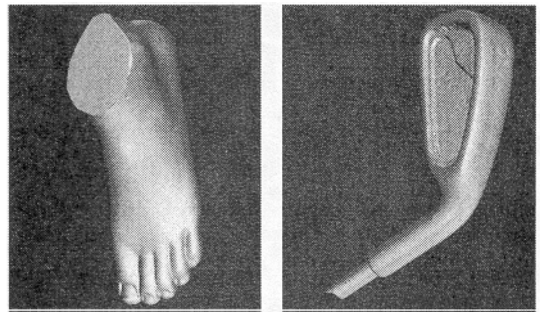

Figure 5: Foot, 50,341 triangles,153 seconds; Club, 33,692 triangles, 122 seconds.

7 Conclusions

We simplified the crust algorithms of [1]. Similar to the original algorithm in [1], our algorithm computes an approximation of a surface from its sample points with the guarantee of homeomorphism and geometric proximity. The merits of our algorithm over the crust algorithm are: (i) it requires only one Voronoi diagram computation as opposed to two such computations in [1]; (ii) it collects a possible set of triangles from the Delaunay triangulation by checking a single condition and thus eliminates normal trimming step of [1]; (iii) the proofs are simpler than [1].

We proved that the output of our algorithm is homeomorphic to the input surface S, assuming that the input surface is smooth, compact and sampled sufficiently densely. We should note, however, that in practice many surfaces have sharp corners and boundaries, so that this result is mainly of theoretical interest.

Our theory is supported, however, by the output of our program on some reasonably large data sets.

Important goals that remain in this area are to correctly reconstruct surfaces with sharp edges and corners, to develop reconstruction algorithms that gracefully handles noise, and to find more efficient algorithms that avoid computing the Delaunay triangulation of all input samples.

Acknowledgement

The authors thank J. Havlicek and R. Wenger for giving insightful comments on the paper.

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