# On The Discrete Scale-Space Formulation

Ji-Young Lim

Academic advisor:

Prof. Dr.-Ing. H. Siegfried Stiehl

Universität Hamburg

Fachbereich Informatik

Arbeitsbereich Kognitive Systeme

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# Abstract

The continuous linear scale-space theory provides a unique framework for visual front-end processes. In practice, a discrete scale-space formulation is necessary since (higher dimensional) discrete signals must be dealt with. The discrete scale-space theory was considered first by Lindeberg. In his work, the derivation of the 1-D discrete scale-space formulation is theoretically obvious. In applying that formulation to higher dimensional discrete signals, however, there exist several open problems and thus the higher dimensional discrete scale-space theory was not fully derived. We review Lindeberg's work in detail in order to understand fundamentals of its formulation and to reveal some unclear aspects. We propose here an improved discrete scale-space formulation for 2-D and 3-D signals based on a few assumptions. Moreover we investigate how to determine the value of a free parameter which plays the role of preserving the rotational symmetry in the higher dimensional discrete scale-space kernel.



# Zusammenfassung

Die kontinuierliche lineare Skalenraum-Theorie stellt den Rahmen für visuelle Frontende-Prozesse dar. In der Praxis müssen jedoch multidimensionale diskrete Signale behandelt werden, wofür eine diskrete Skalenraum-Formulierung für multidimensionale Signale benötigt wird. Der Ansatz zur diskreten Skalenraum-Formulierung nach Lindeberg wird bewertet. Es wird herausgearbeitet, daß die Erweiterung der eindimensionalen diskreten Skalenraum-Formulierung auf den multidimensionalen Fall nicht eindeutig ist. Basierend auf der detaillierten Analyse der diskreten Skalenraum-Theorie nach Lindeberg, wird ein verbesserter Ansatz zur diskreten Skalenraum-Formulierung für zwei- und dreidimensionale Signale entwickelt. Ein wesentlicher Beitrag ist dabei die Bestimmung eines kritischen freien Parameters zur Erhaltung der Rotationsymmetrie im multidimensionalen Fall, die in den Arbeiten von Lindeberg nicht geklärt worden ist.

# Contents

# 1 Introduction

Since Witkin [15] proposed a new way of signal representation by describing zero-crossings of a signal across scales, the scale-space theory has been theoretically developed further by Koenderink [9], Lindeberg [11], [12], and others. The scale-space theory focuses on the basic fact that signal structures exist as meaningful entities only over a certain range of scales. From a given signal one can generate a family of derived signals by successively removing fine-detailed structures when deriving from finer scales to coarser ones. The behavior of structure as scale changes can be analytically described based on a precise mathematical definition. The essence of the results from the scale-space theory is, if one assumes that the first stages of signal processing should be as uncommitted as possible and should not have any *a priori* knowledge about the world from which the signal stems, that the convolution of the initial signal with the Gaussian kernel and its $n$-th order derivatives of different scale is singled out as a canonical class of computational low-level processes (see e.g., [9]).

When we have to use the Gaussian kernel with respect to the scale-space theory, which is formulated for continuous signals whereas in practice we have to deal with discrete signals, a sampled Gaussian is commonly used without mathematical accuracy. In Lim [10], we closely investigated the sampling process of the Gaussian kernels (in 1-D as well as in 2-D and in 3-D) in detail, and derived the measure of difference between the continuous Gaussian and the sampled Gaussian expressed as a formula of the scale parameter. From this investigation, it becomes clear that a sampled Gaussian with a small scale value is not appropriate for approximating the continuous Gaussian. Accordingly, it is indispensable to consider the matter of how to approach the problems caused by discretization of the continuous scale-space theory.

With respect to undesirable effects of a sampled Gaussian applied to the continuous scale-space theory, Lindeberg ([11], [12]) first considered the discrete scale-space theory, and proposed a formulation of the scale-space for discrete signals. In his work, the 1-D discrete scale-space formulation is based on a clear theoretical foundation. However, the theory as related to the extension to higher dimensions[1] is not as simple as in the 1-D case; there exist several open problems in applying that formulation to higher dimensional discrete signals, an thus it was not fully derived.

We closely review Lindeberg's work with respect to the discrete scale-space formulation in Section 2, from which we understand fundamentals of its formulation and reveal some unclear aspects. Motivated by these unclear aspects, in Sections 3 we propose an improved discrete scale-space formulation for 2-D and 3-D signals based on a few assumptions. Finally, we give a conclusion and discuss our future work in Section 4.

[1] Let us use this term for the dimensions higher than 1-D.

# 2 Survey of Lindeberg's Discrete Scale-Space Formulation

Let us begin with the basic assumptions which must hold for a scale-space for discrete signals (Lindeberg [12, p. 63]):

- Every scale-space representation should be generated by a linear and shift-invariant transformation of the original signal. This means that the generation of the scale-space representation can be expressed by the convolution transformation.
- An increasing value of the scale parameter $t$ should correspond to coarser levels of scale and thus to signals with less detailed structure. In particular, $t = 0$ should represent the original signal.
- All signals should be real-valued function defined on the same infinite grid, or in other words, no pyramid representation[2] will be used.

## 2.1 1-D Discrete Scale-Space Formulation

A scale-space kernel for 1-D signals is defined:

**Definition** 1 (DISCRETE SCALE-SPACE KERNEL : 1-D)
*"A 1-D discrete kernel $K : \mathbb{Z} \to \mathbb{R}$ is said to be a scale-space kernel if for all signals $f_{in} : \mathbb{Z} \to \mathbb{R}$ the number of local extrema in the convolved signal $f_{out} = K * f_{in}$ does not exceed the number of local extrema in the original signal."*[3]

Definition 1 implies that the number of local extrema (or, equivalently, zero-crossings) of the convolved signal cannot be greater than the number of local extrema of the original signal.

[2] A pyramid representation of a signal is a set of successively smoothed and sub-sampled representations of the original signal organized in such a way that the number of pixels decreases with a constant factor (usually $2^N$ for a N-D signal) from one layer to the next ([12, p.33]).

[3] A phrase quoted from Lindeberg is cited by the double quotation mark in this section.

### 2.1.1 Kernel Properties

To satisfy the requirement in Definition 1, some kernel properties have to be imposed on the scale-space kernel.

**Proposition** 2 (POSITIVITY)
*"All coefficients of a scale-space kernel must have the same sign: It can be restricted that all coefficients be positive where all $K(n) \geq 0$."*

**Proposition** 3 (UNIMODALITY)
*"The sequence of coefficients of the scale-space kernel $\{K(n)\}_{n=-\infty}^{\infty}$ must be unimodal[4]."*

To prove the necessity of Positivity and Unimodality for a scale-space kernel, let us take a simple example such that the input function is the discrete delta function given by

$$\delta(x) = \begin{cases} 1, & \text{if } x = 0 \\ 0, & \text{otherwise} \end{cases}.$$

In this case, the result of convolving $\delta(x)$ with the discrete scale-space kernel $K(x)$ is equivalent to $K(x)$, since the input signal $\delta(x)$ has only one local maximum and no zero-crossings. This implies that $K(x)$ must not have more than one extremum and no zero-crossing according to Definition 1. Only one local extremum of $K(x)$ satisfies the property of Unimodality in Proposition 3, and having no zero-crossings implies that $K(x)$ fulfills the property of Positivity. Accordingly, it can be said that the discrete scale-space kernel must be positive and unimodal.

Let us consider a two-kernel[5] as the simplest discrete scale-space kernel which satisfies the above two properties:

$$K^2(n) = \begin{cases} p, & \text{if } n = 0 \\ q, & \text{if } n = -1 \\ 0, & \text{otherwise} \end{cases}, \tag{2.1}$$

[4]A real sequence is said to be unimodal if it is first ascending (descending) and then descending (ascending).

[5]Let us denominate the kernel which has $N$ non-zero kernel coefficients a $N$-kernel.

where $p \geq 0$, $q \geq 0$ and $p + q = 1$. In this case, it is guaranteed that the number of zero-crossings (local extrema) in $f_{out} = K^2 * f_{in}$ can not exceed the number of zero crossings in $f_{in}$ (see Figure 3.4. in [12, p. 67]). Therefore, it can be said that any kernel of the form (2.1) is a discrete scale-space kernel. Using the fact that, if two kernels $K_a$ and $K_b$ are scale-space kernels, then $K_a * K_b$ is also a scale-space kernel, we can obtain another property of the discrete scale-space kernel:

**Proposition** 4 (Repeated Averaging and Scale-Space Kernels)
*"All kernels $K$ of the form $*_{i=1}^{n} K_i^2$ are discrete scale-space kernels."*[6]

It is noticeable that coefficients of the filter generated by Proposition 4 can be regarded as generalized binomial coefficients[7] which can be generated by a generating function.

Some special mathematical functions may be defined by generating function[8]. Since generating functions play a very important role in order to sustain the theoretical bases of the discrete scale-space formulation, it is worth looking into its fundamentals (see e.g., [1], [4], [5], or [14]): An infinite sequence $< a_0, a_1, a_2, ... >$ that we deal with can conveniently be represented as a *power series* in an auxiliary variable $z$,

$$A(z) = a_0 + a_1 z + a_2 z^2 + ... = \sum_{k \geq 0} a_k z^k.$$

It is appropriate to use the letter $z$ for the auxiliary variable, because $z$ is often thought as a complex number. A generating function is useful since it is a single quantity that represents an entire infinite sequence. If $A(z)$ is any power series $\sum_{k \geq 0} a_k z^k$, it is convenient to write

$$[z^n]A(z) = a_n;$$

in other words, $[z^n]A(z)$ denotes the coefficient of $z^n$ in $A(z)$. Let $A(z)$ be the generating function for $< a_0, a_1, a_2, ... >$ and $B(z)$ the one for another sequence $< b_0, b_1, b_2, ... >$. Then the product $A(z)B(z)$ is the power series given as

$$\begin{aligned} &(a_0 + a_1 z + a_2 z^2 + ...)(b_0 + b_1 z + b_2 z^2 + ...) \\ &= a_0 b_0 + (a_0 b_1 + a_1 b_0) z + (a_0 b_2 + a_1 b_1 + a_2 b_0) z^2 + .... \end{aligned}$$

[6] $*_{i=1}^{n} K_i^2 = \underbrace{K^2 * \cdots * K^2}_{n}$.

[7] The ordinary binomial coefficients are obtained as a special case if all $p_i$ and $q_i$ are equal.

[8] It is also referred to as *z-transform*.

The coefficient of $z^n$ in this product is

$$a_0 b_n + a_1 b_{n-1} + ... + a_n b_0 = \sum_{k=0}^{n} a_k b_{n-k}.$$

If we express this sum as the general form

$$c_n = \sum_{k=0}^{n} a_k b_{n-k},$$

we have

$$c_n = [z^n] A(z) B(z).$$

The sequence $< c_n >$ is the result of the convolution of the sequences $< a_n >$ and $< b_n >$. This implies that convolution of sequences corresponds to multiplication of generating functions.

**Proposition** 5 (Generating Functions of Generalized Binomial Kernels)
*"All kernels with the generating function*

$$\varphi_K(z) = \sum_{n=-\infty}^{\infty} K(n) z^n$$

*of the form*

$$\varphi_K(z) = c z^k \prod_{i=1}^{N} (p_i + q_i z),$$

*where* $p_i > 0$*,* $q_i > 0$ *and* $k \in \mathbb{Z}$*, are discrete scale-space kernels."*

Proposition 5 can be proved by showing the generating functions of kernels: The generating function of a kernel of the form (2.1) is given as

$$\varphi_{K_i^{(2)}}(z) = p_i + q_i z,$$

and the convolution of kernels in the spatial domain[9] corresponds to the multiplication of their generating functions. Therefore, Proposition 4 can be rewritten without changing

[9] In this report, we skip to explain the properties of kernels in frequency domain. See [11, Sec. III] and [12, p. 70-74] for details.

the scale-space properties as

$$\begin{aligned}\varphi_h(z) &= \varphi_{K_1^{(2)}}(z)\varphi_{K_2^{(2)}}(z)\varphi_{K_3^{(2)}}(z)...\varphi_{K_N^{(2)}}(z),\\ \varphi_K(z) &= cz^k\varphi_h(z),\end{aligned}$$

where a constant scaling factor $c$ or a translation $z^k$ does not affect the number of local extrema.

Now, let us get acquainted with a few termini which are necessary to comprehend the kernel classifying process with respect to discrete scale-space kernels:

- A Toeplitz matrix :

  If the convolution transformation $f_{out} = K * f_{in}$ is represented in the matrix form $f_{out} = Cf_{in}$, a matrix with constant values along the diagonals $C_{i,j} = K(i-j)$ results. Such a matrix is called a *Toeplitz* matrix. Let $K : \mathbb{Z} \to \mathbb{R}$ be a discrete kernel with finite support and filter coefficients $c_n = K(n)$. For some dimension $N$, the $N \times N$ convolution matrix is expressed as

$$C^{(N)} = \begin{pmatrix} c_0 & c_{-1} & ... & c_{2-N} & c_{1-N} \\ c_1 & c_0 & c_{-1} & ... & c_{2-N} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ c_{N-2} & ... & c_1 & c_0 & c_{-1} \\ c_{N-1} & c_{N-2} & ... & c_1 & c_0 \end{pmatrix}.$$

- Minors of matrix ([12, p. 76]) :

  Given a kernel $K : X \times Y \to \mathbb{R}$, one can form *minors* of arbitrary order $r$ by selections of $x_1 < x_2 < ... < x_r$ from $X \subset \mathbb{Z}$ and $y_1 < y_2 < ... < y_r$ from $Y \subset \mathbb{Z}$. The determinant of the resulting matrix with components $\{K(x_i, y_i)\}_{i,j=1,...,r}$ is called "a minor of order r" and denoted by

$$K\begin{pmatrix} x_1, & x_2, & ..., & x_r \\ y_1, & y_2, & ..., & y_r \end{pmatrix} = \begin{vmatrix} K(x_1,y_1) & K(x_1,y_2) & .. & K(x_1,y_r) \\ K(x_2,y_1) & K(x_2,y_2) & .. & K(x_2,y_r) \\ . & . & & . \\ . & . & & . \\ K(x_r,y_1) & K(x_r,y_2) & .. & K(x_r,y_r) \end{vmatrix}.$$

- Pólya frequency sequence :

  A sequence $\{c_n\}_{n=-\infty}^{\infty}$ is said to be a *Pólya frequency sequence* if all minors of the infinite Toeplitz matrix are non-negative. Especially provided that its generating function $\varphi(z) = \sum_{n=-\infty}^{\infty} c_n z^n \neq 0$ converges in an annulus $r < |z| < R$ $(0 < r < 1 < R)$, it is called *a normalized Pólya frequency sequence.* An infinite sequence $\{c_n\}_{n=-\infty}^{\infty}$ is a Pólya frequency sequence if and only if its generating function $\varphi(z) = \sum_{n=-\infty}^{\infty} c_n z^n$ is of the form

$$\varphi_K(z) = cz^k e^{(q_{-1}z^{-1}+q_1 z^1)} \prod_{i=1}^{\infty} \frac{(1+\alpha_i z)(1+\delta_i z^{-1})}{(1-\beta_i z)(1-\gamma_i z^{-1})}, \tag{2.2}$$

  where $c > 0, k \in \mathbb{Z}, q_{-1}, q_1, \alpha_i, \beta_i, \delta_i, \gamma_i \geq 0$, and $\sum_{i=1}^{\infty}(\alpha_i + \beta_i + \delta_i + \gamma_i) < \infty$. The sequence $\{c_n\}_{n=-\infty}^{\infty}$ is normalized if and only if in addition it holds that $\beta_i < 1$ and $\gamma_i < 1$.

Lindeberg showed that a discrete kernel $K : \mathbb{Z} \to \mathbb{R}$ is a scale-space kernel if and only if the corresponding sequence of kernel coefficients $\{K(n)\}_{n=-\infty}^{\infty}$ is a normalized Pólya frequency sequence (see [12, p. 76-81] for its proof).

In particular for kernels with finite support, (2.2) is reduced for some finite $N$ to

$$\varphi_K(z) = cz^k \prod_{i=1}^{N} (1+\alpha_i z)(1+\delta_i z^{-1}),$$

where $c$ is a rescaling factor and $z^k$ is a translation factor. $(1+\alpha_i z)$ and $(1+\delta_i z^{-1})$ can be easily recognized as the rewritten versions of the generating functions of a two-kernel. According to Proposition 4, the kernels of the form $*_{i=1}^{N} K_i^2$ are the only discrete scale-space kernels with finite support, which implies that the convolution with a finite scale-space kernel can be decomposed into a convolution with kernels having two strictly positive consecutive filter coefficients.

### 2.1.2 Discrete Scale-Space Construction

Given any 1-D signal $f : \mathbb{Z} \to \mathbb{R}$, let us assume that the scale-space representation $L : \mathbb{Z} \times \mathbb{R}_+ \to \mathbb{R}$ should be generated by convolution of $f$ with a one-parameter family of kernels $T : \mathbb{Z} \times \mathbb{R}_+ \to \mathbb{R}$ such that $L(x;0) = f(x)$ and

$$L(x;t) = \sum_{n=-\infty}^{\infty} T(n;t) f(x-n), \qquad t > 0 \tag{2.3}$$

hold, where each kernel $T(n;t)$ is a scale-space kernel, i.e., $T(n;t)$ satisfies

- the semi-group property given by

  $$T(\cdot;s) * T(\cdot;t) = T(\cdot;s+t),$$

  which ensures that the scale-space property described in Definition 1 holds between two levels of scale, and

- the normalization criterion

  $$\sum_{n=-\infty}^{\infty} T(n;t) = 1.$$

As mentioned in the previous section, a discrete kernel is a scale-space kernel if and only if the corresponding sequence of kernel coefficients is a normalized Pólya frequency sequence. With respect to the semi-group property, the theorem of Karlin ([12, p. 85]) states that the only semi-group of normalized Pólya frequency sequences has a generating function of the form

$$\varphi(z) = e^{t(az+bz^{-1})}, \tag{2.4}$$

where $t > 0$ and $a, b \geq 0$.

Then, it can be shown that the generating function of a discrete scale-space kernel of the form (2.4) can be derived from (2.2): If a family $h(\cdot;t)$ possesses the semi-group property $h(\cdot;s) * h(\cdot;t) = h(\cdot;s+t)$, its generating function must obey $\varphi_{h(\cdot;s)} \cdot \varphi_{h(\cdot;t)} = \varphi_{h(\cdot;s+t)}$ for all non-negative $s$ and $t$. This respect excludes the factors $z^k$, $(1+\alpha_i z)$, $(1+\delta_i z^{-1})$, $(1-\beta_i z)$, and $(1-\gamma_i z^{-1})$ from (2.2); what remains are the constant and the exponential factors. The argument of the exponential factor must be linear in $t$ in order to fulfill the semi-group property of the kernels under convolution. Furthermore, due to the symmetry the generating function must satisfy $\varphi_h(z^{-1}) = \varphi_h(z)$, which leads to $a = b$ (for simplicity, $a = b = \frac{\alpha}{2}$) in (2.4). Consequently the generating function of the form

$$\varphi_t(z) = e^{\frac{\alpha t}{2}(z+z^{-1})} \tag{2.5}$$

is obtained, which corresponds to the generating function for the modified Bessel functions of integer order. Bessel functions and closely related functions form a rich area of mathematical analysis with many representations and useful properties ([1]). Although Bessel functions are primary solutions of differential equations, it is instructive and convenient to derive them from the generating function. Let us consider a function of two variables given as

$$\varphi(x, z) = e^{\frac{x}{2}(z-z^{-1})},$$

then we can obtain

$$e^{\frac{x}{2}(z-z^{-1})} = \sum_{n=-\infty}^{\infty} J_n(x) z^n,$$

where the coefficient of $z^n$, $J_n(x)$, is defined to be *a Bessel function of the first kind of integer order* $n$. In other words, $e^{\frac{x}{2}(z-z^{-1})}$ is the generating function for the Bessel function of the first kind of integer order. We can prove the equation above using MacLaurin series: The MacLaurin series of $e^x$ is given as

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \ldots = \sum_{n=0}^{\infty} \frac{x^n}{n!}.$$

Then we can expand the exponentials as a product of MacLaurin series in $\frac{xz}{2}$ and $-\frac{x}{2z}$, respectively,

$$e^{\frac{xz}{2}} \cdot e^{-\frac{x}{2z}} = \sum_{r=0}^{\infty} \left(\frac{x}{2}\right)^r \frac{z^r}{r!} \sum_{s=0}^{\infty} (-1)^s \left(\frac{x}{2}\right)^s \frac{z^{-s}}{s!}.$$

By introducing $n$ $(n = r - s)$ for a given $s$,

$$\begin{aligned}\sum_{n+s=0}^{\infty} \left(\frac{x}{2}\right)^{n+s} \frac{z^{n+s}}{(n+s)!} \sum_{s=0}^{\infty} (-1)^s \left(\frac{x}{2}\right)^s \frac{z^{-s}}{s!} &= \sum_{n=-\infty}^{\infty} \sum_{s=0}^{\infty} \frac{(-1)^s}{s!(n+s)!} \left(\frac{x}{2}\right)^{n+2s} z^n \\ &= \sum_{n=-\infty}^{\infty} J_n(x) z^n.\end{aligned}$$

In an analogous way, we can get the generating function for *a modified Bessel function of integer order* $n$, $I_n(x)$,

$$e^{(\frac{x}{2})(z+z^{-1})} = \sum_{n=-\infty}^{\infty} I_n(x) z^n.$$

In terms of infinite series the modified Bessel function is equivalent to removing the $(-1)^s$ sign term of the Bessel function and writing

$$I_n(x) = \sum_{s=0}^{\infty} \frac{1}{s!(s+n)!} \left(\frac{x}{2}\right)^{2s+n},$$
$$I_{-n}(x) = \sum_{s=0}^{\infty} \frac{1}{s!(s-n)!} \left(\frac{x}{2}\right)^{2s-n}.$$

For integer $n$ this yields

$$I_n(x) = I_{-n}(x).$$

Returning back to the discrete scale-space kernel, (2.5) is then expressed using the generating function of the modified Bessel functions of integer order as

$$\varphi_t(z) = \sum_{n=-\infty}^{\infty} I_n(\alpha t) z^n.$$

Since

$$\sum_{n=-\infty}^{\infty} I_n(\alpha t) = e^{\alpha t}$$

holds, a normalized kernel is obtained if $T : \mathbb{Z} \times \mathbb{R}_+ \to \mathbb{R}$ is defined by

$$T(n; t) = e^{-\alpha t} I_n(\alpha t),$$

where $e^{-\alpha t}$ is the very constant which remains to be found in (2.4). The semi-group property is trivially preserved after normalization.

The kernel $T(n; t) = e^{-\alpha t} I_n(\alpha t)$ possesses similar properties in the discrete case as those which make the ordinary Gaussian kernel special in the continuous case. Therefore, it is natural to refer to it as *the discrete analogue of the Gaussian kernel*:

**Definition** 6 (Discrete Analogue of the Gaussian Kernel)
*"The kernel $T : \mathbb{Z} \times \mathbb{R} \to \mathbb{R}$ given by $T(n;t) = e^{-\alpha t} I_n(\alpha t)$[10] is called the discrete analogue of the Gaussian kernel, or, the discrete Gaussian."*

When the transformation of the form (2.3) is implemented according to Definition 6, a few numerical problems come about ([11, Sec. VI, p. 240]):

- The infinite convolution sum must be replaced by a finite one.
- Normally the modified Bessel functions are not available as standard library routines. Therefore, it is required to design an algorithm to generate coefficients of the filter $T(n;t)$ for a given value of $t$.
- To deal with a realistic finite signal it is unavoidable to make a finite approximation of (2.3), which may cause any truncation error.

These problems could give rise to further net effects by approximation. Alternatively, instead of using an infinite convolution, the scale-space representation can be constructed by the discretized diffusion equation (see the below sections).

In the following sections, it will be shown on one hand that the scale-space representation based on the discrete analogue Gaussian can be constructed by the discretized diffusion equation. On the other hand, inversely, it will be seen how the discretized diffusion equation satisfies the properties of the scale-space representation generated by discrete scale-space kernels.

**Discretized Diffusion Equation** The discrete scale-space representation given by (2.3) can be interpreted in terms of a discretized version of the diffusion equation. By the recurrence relation for the modified Bessel functions

$$\partial_t I_n(t) = \frac{I_{n-1}(t) + I_{n+1}(t)}{2},$$

[10] From now on, let us assume $\alpha = 1$.

it can be easily shown that $T(n;t) = e^{-t} I_n(t)$ satisfies

$$\begin{aligned}\partial_t T(n;t) &= e^{-t}\partial_t I_n(t) - e^{-t} I_n(t) \\ &= \frac{e^{-t}}{2}\left[I_{n-1}(t) + I_{n+1}(t)\right] - e^{-t} I_n(t) \\ &= \frac{1}{2}\left[T(n-1;t) + T(n+1;t)\right] - T(n;t) \\ &= \frac{1}{2}\left[T(n-1;t) - 2T(n;t) + T(n+1;t)\right],\end{aligned}$$

and therefore

$$\begin{aligned}\partial_t L(x;t) &= \partial_t \sum_{n=-\infty}^{\infty} T(n;t) f(x-n) \\ &= \sum_{n=-\infty}^{\infty} \partial_t T(n;t) f(x-n) \\ &= \frac{1}{2}\left[L(x-1;t) - 2L(x;t) + L(x+1;t)\right]\end{aligned}$$

holds. This result can be further discretized with respect to scale $t$ using Euler's method[11] such that

$$L_i^{k+1} = L_i^k + \triangle t\left(\partial_t L_i^k\right),$$

which gives the iteration formula

$$\begin{aligned}L_i^{k+1} &= L_i^k + \triangle t\left(\frac{1}{2}L_{i+1}^k - L_i^k + \frac{1}{2}L_{i-1}^k\right) \\ &= \frac{1}{2}\triangle t L_{i+1}^k + (1-\triangle t)L_i^k + \frac{1}{2}\triangle t L_{i-1}^k,\end{aligned} \tag{2.6}$$

where the subscripts denote the spatial coordinates and the superscripts represent the iteration indices.

[11] It is the approximation of the derivative by the difference quotient for explicit differential equations of the first order. According to Euler's method,

$$\begin{aligned}y' = f(x,y) \quad &\to \frac{y(x_i+h)-y(x_i)}{h} \approx f(x_i, y(x_i)) + O(h) \\ &\to y(x_i+h) = y(x_i) + f(x_i, y(x_i)) \cdot h,\end{aligned}$$

where $h$ is the step size ($h = |x_{i+1} - x_i|$) and $O(h)$ is the error of approximation ([6, p. 676-677]).

Equivalently, iteration with the formula (2.6) can be described as discrete convolution with the three-kernel given by

$$\left( \frac{1}{2}\triangle t \quad 1 - \triangle t \quad \frac{1}{2}\triangle t \right). \tag{2.7}$$

Lindeberg proved that a three-kernel with positive elements $c_{-1}$, $c_0$ and $c_1$ is a scale-space kernel if and only if $c_0^2 \geq 4c_{-1}c_1$ (see [11, p. 238] for the proof). That is to say, this kernel is a scale-space kernel if and only if

$$(1 - \triangle t)^2 \geq 4 \left( \frac{1}{2}\triangle t \right)^2,$$

which leads to

$$\triangle t \leq \frac{1}{2}.$$

**Proposition 7** (Diffusion Equation and Discrete Scale-Space Kernels)
*"All symmetric discrete scale-space kernels with finite support arise from repeated application of the discretization of the diffusion equation (2.6), using if necessary different* $\triangle t \in \left[0, \frac{1}{2}\right]$*."*

As mentioned, there exist a few problems to implement (2.3) as it is. However, Proposition 7 makes it practically possible by repeatedly applying the discretized diffusion equation with the discrete scale parameter satisfying $\triangle t \in \left[0, \frac{1}{2}\right]$.

**Sufficiency of the Discretized Diffusion Equation** By deriving the generating function of the three-kernel of the form (2.7), it can be shown that discretization of (2.6) with respect to the scale with $n$ steps (i.e., the step size $\triangle t = t/n$ satisfying $\triangle t \leq \frac{1}{2}$) is theoretically sufficient to construct the scale-space representation. The generating function for the three-kernel of the form (2.7) is

$$\varphi_{step}(z) = \frac{1}{2}\triangle t z^{-1} + (1 - \triangle t) + \frac{1}{2}\triangle t z,$$

where the subscript *step* means one iteration. The final solution of the discretized diffusion equation can be obtained by convolution with the composed kernel $K_{composed} = *_{i=1}^{n} K_{step}$

on the basis of the semi-group property, and furthermore the convolution of kernels corresponds to the multiplication of generating functions. Thus, the generating function of the composed kernel can be obtained as

$$\begin{aligned}\varphi_{composed,n}(z) &= \prod_{i=1}^{n} \varphi_{step}(z) \\ &= \left(\frac{1}{2}\triangle t z^{-1} + (1-\triangle t) + \frac{1}{2}\triangle t z\right)^n \\ &= \left(1 + \triangle t\left(\frac{z^{-1}}{2} - 1 + \frac{z}{2}\right)\right)^n \\ &= \left(1 + \frac{t}{n}\left(\frac{z^{-1}}{2} - 1 + \frac{z}{2}\right)\right)^n.\end{aligned}$$

Let us here use the fact $\lim_{n\to\infty}(1+\alpha_n/n)^n = e^{\alpha}$ if $\lim_{n\to\infty}\alpha_n = \alpha$; since

$$\lim_{n\to\infty} t\left(\frac{z^{-1}}{2} - 1 + \frac{z}{2}\right) = t\left(\frac{z^{-1}}{2} - 1 + \frac{z}{2}\right),$$

it follows that

$$\begin{aligned}\lim_{n\to\infty}\varphi_{composed,n}(z) &= e^{t\left(\frac{z^{-1}}{2}-1+\frac{z}{2}\right)} \\ &= e^{-t}e^{t(z^{-1}+z)/2}.\end{aligned} \tag{2.8}$$

It is clear that $e^{t(z^{-1}+z)/2}$ in (2.8) corresponds to the generating function for modified Bessel functions of integer order, and accordingly $e^{-t}e^{t(z^{-1}+z)/2}$ is the generating function ($\alpha = 1$) of the family of discrete scale-space kernels by Definition 6.

Consequently we regard this result as the proof of the property that the transformation (2.3) obtained by convolution of a discrete signal with the discrete analogue of the Gaussian is equivalent to the analytical solution of the system equations obtained by discretizing the diffusion equation. In this context, Lindeberg concluded that the natural way to apply the scale-space theory to discrete signals is by discretizing the diffusion equation, not the convolution integral.

## 2.2 N-D Discrete Scale-Space Formulation

The scale-space representation for higher-dimensional discrete signals can be constructed analogously to the 1-D case. While the 1-D scale-space theory for discrete signals is theoretically obvious, its extension to higher[12] dimension is not so clear as the 1-D case since there are no non-trivial kernels with the property that they never introduce new local extrema. Whereas the number of local extrema in one dimension is a natural measure of structure, in higher dimensions the problem that new local extrema (zero-crossings) can be created by linear smoothing is inherent and inescapable (see [11, Fig. 4]).

Bearing in mind this difficulty, in this section we look into the scale-space formulation for N-D discrete signals proposed by Lindeberg.

### 2.2.1 Non-Enhancement Requirement

In case of 2-D continuous signals, the scale-space representation can be derived on three assumptions, namely causality, homogeneity, and isotropy (see [9]). Using differential geometry, the diffusion equation (or equivalently the convolution with the Gaussian kernel) fulfills the requirement of the scale-space representation ([9]). However, it is impossible to relate these assumptions directly to discrete signals since no direct correspondences to level curves or to differential geometry described for continuous signals exist in the discrete domain.

With respect to discrete signals, alternatively, Lindeberg restricted the causality to obey the non-enhancement requirement, i.e., if for some scale level $t_0$ a point $x_0$ has a local maximum (minimum) in the scale-space representation at that level (regarded as a function of the space coordinates only), then its value must not increase (decrease) when the scale parameter increases (decreases). That is to say, local extrema should not be enhanced when the scale parameter is continuously increased.

On the basis of the non-enhancement requirement, let us assume that for a given spatially discrete signal $f : \mathbb{Z}^N \to \mathbb{R}$ the scale-space representation is generated by convolution

[12] In this report, we only consider the 2-D and 3-D cases.

with a one-parameter family of kernels of the form

$$L(x;t) = \sum_{\xi\in\mathbb{Z}^N} T(\xi;t) f(x-\xi), \tag{2.9}$$

where $T : \mathbb{Z}^N \times \mathbb{R}_+ \to \mathbb{R}$ in $l_1$[13] satisfies

- $T(\cdot;0) = \delta(\cdot)$,
- the semi-group property, i.e., $T(\cdot;s) * T(\cdot;t) = T(\cdot;s+t)$,
- the symmetry constraints, i.e., $T(-x_1, x_2, ..., x_N; t) = T(x_1, x_2, ..., x_N; t)$ and $T(P_k^N(x_1, x_2, ..., x_N); t) = T(x_1, x_2, ..., x_N; t)$ for all $x = (x_1, x_2, ..., x_N) \in \mathbb{Z}^N$, all $t \in \mathbb{R}_+$, and all possible permutations $P_k^N$ of $N$ elements, and
- the continuity requirement, i.e., $||T(\cdot;t) - \delta(\cdot)||_1 \to 0$ when $t \downarrow 0$.

Provided that the input signal $f$ is sufficiently regular, these conditions for the family of kernels $T$ guarantee that the representation $L$ is differentiable and satisfies a system of linear differential equations, i.e., $L$ obeys $\partial_t L = AL$ for some linear operator $A$ (see [12, Lemma 4.5. (p. 107)] for its proof).

According to Lindeberg ([12, p. 104]), for a given point $x \in \mathbb{Z}^N$ its neighborhood $N(x)$ is defined as

$$N(x) = \{\xi \in \mathbb{Z}^N : (||x-\xi||_\infty \le 1) \wedge (\xi \ne x)\}, \tag{2.10}$$

and extremum points are characterized as

- Discrete local maxima :
  A point $x \in \mathbb{Z}^N$ is said to be a (weak) local maximum of a function $g : \mathbb{Z}^N \to \mathbb{R}$ if $g(x) \ge g(\xi)$ for all $\xi \in N(x)$.

[13] By definition ([7]),

$$l_p(\text{or}, l^p) = \left\{ \{f(k)\}_{k\in\mathbb{Z}} | \sum_{k=-\infty}^{\infty} |f(k)|^p < \infty \right\}.$$

- Discrete local minima :

  A point $x \in \mathbb{Z}^N$ is said to be a (weak) local minimum of a function $g : \mathbb{Z}^N \to \mathbb{R}$ if $g(x) \leq g(\xi)$ for all $\xi \in N(x)$.

Using these basic definitions it can be said that a differentiable one-parameter family of discrete signals $L : \mathbb{Z}^N \times \mathbb{R}_+ \to \mathbb{R}$ does not enhance local extrema for every value of the scale parameter $t_0 \in \mathbb{R}_+$, such that if $x_0 \in \mathbb{Z}^N$ is a local extremum point for the mapping $x \mapsto L(x; t_0)$ then the derivative of $L$ with respect to $t$ in this point satisfies

- $\partial_t L(x_0; t_0) \leq 0$ if $x_0$ is a local maximum point,
- $\partial_t L(x_0; t_0) \geq 0$ if $x_0$ is a local minimum point.

Consequently the scale-space representation can be defined as follow:

**Definition** 8 (SCALE-SPACE REPRESENTATION)
*"A scale-space representation $L : \mathbb{Z}^N \times \mathbb{R}_+ \to \mathbb{R}$ of a signal $f : \mathbb{Z}^N \to \mathbb{R}$ generated by a family of kernels $T : \mathbb{Z}^N \times \mathbb{R}_+ \to \mathbb{R}$, which satisfy the non-enhancement requirement, is said to be a scale-space representation of $f$."*

### 2.2.2 The Semi-Discretized Diffusion Equation

As seen in Section 2.1.2, since convolution of a 1-D discrete signal with the 1-D discrete analogue of the Gaussian is equivalent to the analytical solution of the discretized diffusion equation, the natural way to apply the scale-space theory to 1-D discrete signals is by discretizing the diffusion equation. For higher dimensional signals, the scale-space representation can be constructed by a semi-discretized diffusion equation in a similar manner analogously to the 1-D case. The evolution of the scale-space representation of N-D discrete signals over scales can be described by a semi-discretized diffusion equation which is formulated by the infinitesimal scale-space generator.

In the following sections, we describe the concept of the infinitesimal scale-space generator with respect to the semi-discretized diffusion equation. Then we analyze the effect of the parameter $\gamma$ in the infinitesimal scale-space generator.

**The Infinitesimal Scale-Space Generator** The family of $L$ of the form (2.9) satisfies a semi-discretized[14] version of the diffusion equation

$$\partial_t L = A_{ScSp} L \tag{2.11}$$

for some infinitesimal scale-space generator $A_{ScSp}$. Note that it is possible to derive derivatives of the scale-space representation with respect to the scale parameter using the property of kernels $T$ (see Section 2.2.1).

We can easily envisage that (2.10) corresponds to two-connectivity in the 1-D case, eight-connectivity in the 2-D case and twenty six-connectivity in the 3-D case. This connectivity of the neighborhood enables us to derive the point operator[15], e.g., corresponding to the Laplacian. Then, for a given signal $f : \mathbb{Z}^N \to \mathbb{R}$ the Laplacian for 1-D, 2-D and 3-D are respectively given by

- 1-D case ($N = 1$): $\nabla_3^2$

$$(\nabla_3^2 f)_0 = f_{-1} - 2f_0 + f_1,$$

- 2-D case ($N = 2$): $\nabla_5^2$ and $\nabla_{\times^2}^2$[16]

$$(\nabla_5^2 f)_{0,0} = f_{-1,0} + f_{1,0} + f_{0,-1} + f_{0,1} - 4f_{0,0},$$
$$(\nabla_{\times^2}^2 f)_{0,0} = \frac{1}{2}(f_{-1,-1} + f_{-1,1} + f_{1,-1} + f_{1,1} - 4f_{0,0}) \qquad \text{and}$$

---

[14] The term "semi" seems to be used by Lindeberg due to the indefinite parameter $\gamma$.

[15] In case of a 1-D signal, for example, let $f_{-1}$, $f_0$, and $f_1$, respectively, denote $f(x-1)$, $f(x)$ and $f(x+1)$ for a given point $x \in \mathbb{Z}$.

[16] We cannot find the reason (or account) why Lindeberg inserted the coefficient $\frac{1}{2}$ in defining $\nabla_{\times^2}^2$.

- 3-D case ($N = 3$): $\nabla_7^2$, $\nabla_{+^3}^2$ and $\nabla_{\times^3}^2$[17]

$$\begin{aligned}
(\nabla_7^2 f)_{0,0,0} =& f_{-1,0,0} + f_{1,0,0} + f_{0,-1,0} + f_{0,1,0} + f_{0,0,-1} + f_{0,0,1} - 6f_{0,0,0} \\
(\nabla_{+^3}^2 f)_{0,0,0} =& \frac{1}{4}(f_{-1,-1,0} + f_{-1,1,0} + f_{1,-1,0} + f_{1,1,0} + f_{-1,0,-1} + f_{-1,0,1} + \\
& f_{1,0,-1} + f_{1,0,1} + f_{0,-1,-1} + f_{0,-1,1} + f_{0,1,-1} + f_{0,1,1} - 12f_{0,0,0}), \\
(\nabla_{\times^3}^2 f)_{0,0,0} =& \frac{1}{4}(f_{-1,-1,-1} + f_{-1,-1,1} + f_{-1,1,-1} + f_{1,-1,-1} + f_{-1,1,1} + \\
& f_{1,-1,1} + f_{1,1,-1} + f_{1,1,1} - 8f_{0,0,0}).
\end{aligned}$$

Then (2.11) reduces in 1-D, 2-D, and 3-D, respectively, to

$$\partial_t L = \alpha_1 \nabla_3^2 L, \tag{2.12}$$

$$\partial_t L = \alpha_1 \nabla_5^2 L + \alpha_2 \nabla_{\times^2}^2 L \quad \text{and} \tag{2.13}$$

$$\partial_t L = \alpha_1 \nabla_7^2 L + \alpha_2 \nabla_{+^3}^2 L + \alpha_3 \nabla_{\times^3}^2 L, \tag{2.14}$$

for some constants $\alpha_1 \geq 0$, $\alpha_2 \geq 0$, and $\alpha_3 \geq 0$ (see [12, p. 109-112] for the proof). This implies that a one-parameter family of discrete signals satisfying the differential equation (2.11) obeys the non-enhancement requirement by Definition 8.

$\alpha_1$ in (2.12) is set to $\frac{1}{2}$ (see Section 2.1.2). In the 1-D case, only one factor of the Laplacian ($\nabla_3^2 L$) exists. For the case of higher-dimensions, however, the number of the factors of the Laplacian is proportional to the dimension of the underlying signal, e.g., two factors (for $\nabla_5^2$ and $\nabla_{\times^2}^2$) in the 2-D case and three factors (for $\nabla_7^2$, $\nabla_{+^3}^2$ and $\nabla_{\times^3}^2$) in the 3-D case. Considering the normalization property, (2.13) and (2.14) can be reparametrized by introducing $\gamma_i \in [0, 1]$. Then (2.13) is rewritten in the form[18]

$$\partial_t L = \frac{1}{2}\left((1-\gamma_1)\nabla_5^2 L + \gamma_1 \nabla_{\times^2}^2 L\right) = \frac{1}{2}\nabla_{\gamma_1}^2 L, \tag{2.15}$$

[17] We cannot find any reason (or account) why Lindeberg inserted the coefficients $\frac{1}{4}$ in defining $\nabla_{+^3}^2$ and $\nabla_{\times^3}^2$.

[18] According to the point operators of the Laplacian, it can be expressed as

$$\partial_t L = \frac{1}{2}(1-\gamma_1)\begin{pmatrix} & 1 & \\ 1 & -4 & 1 \\ & 1 & \end{pmatrix} L + \frac{1}{2}\gamma_1 \begin{pmatrix} \frac{1}{2} & & \frac{1}{2} \\ & -2 & \\ \frac{1}{2} & & \frac{1}{2} \end{pmatrix} L \quad .$$

and in a same manner (2.14) is rewritten by means of $\gamma_1$ and $\gamma_2$ as

$$\partial_t L = \frac{1}{2}\left((1-\gamma_1-\gamma_2)\nabla_7^2 L + \gamma_1 \nabla_{+^3}^2 L + \gamma_2 \nabla_{\times^3}^2 L\right) = \frac{1}{2}\nabla_{\gamma_1,\gamma_2}^2 L.$$

It is required from the reparametrization to determine the parameter $\gamma_i$ which are not definite.

**Sufficiency of The Infinitesimal Scale-Space Generator** By deriving the generating function of the infinitesimal scale-space generator $A_{ScSp}$, its sufficiency for describing the scale-space representation can be shown. Note that for simplicity we deal with the 2-D case only. However, further extension to 3-D or other higher dimensions can be done in a similar manner as for the 2-D case.

Let $L : \mathbb{Z}^2 \times \mathbb{R}_+ \to \mathbb{R}$ be the scale-space representation of a discrete signal $f : \mathbb{Z}^2 \to \mathbb{R}$ with the initial condition $L(\cdot;0) = f(\cdot)$. Then (2.15) can be further discretized using Euler's explicit method (compare with (2.6) in Section 2.1.2) with the scale step $\triangle t$;

$$\begin{aligned} L_{i,j}^{k+1} =& L_{i,j}^k + \triangle t\left(\partial_t L_{i,j}^k\right) \\ =& L_{i,j}^k + \triangle t \cdot \frac{1}{2}\left((1-\gamma_1)\nabla_5^2 L + \gamma_1 \nabla_{\times^2}^2 L\right) \\ =& L_{i,j}^k + \triangle t \frac{1}{2}(1-\gamma_1)(L_{i-1,j}^k + L_{i+1,j}^k + L_{i,j-1}^k + L_{i,j+1}^k - 4L_{i,j}^k) \\ & + \triangle t \frac{1}{2}\gamma_1\left(\frac{1}{2}L_{i-1,j-1}^k + \frac{1}{2}L_{i-1,j+1}^k + \frac{1}{2}L_{i+1,j-1}^k + \frac{1}{2}L_{i+1,j+1}^k - 2L_{i,j}^k\right) \\ =& (1-\triangle t(2-\gamma_1))L_{i,j}^k \\ & + \triangle t \frac{1}{2}(1-\gamma_1)(L_{i-1,j}^k + L_{i+1,j}^k + L_{i,j-1}^k + L_{i,j+1}^k) \\ & + \triangle t \frac{1}{4}\gamma_1(L_{i-1,j-1}^k + L_{i-1,j+1}^k + L_{i+1,j-1}^k + L_{i+,j+1}^k). \end{aligned} \tag{2.16}$$

In the iteration formula above, the subscripts $i$ and $j$ denote the spatial coordinates $x$ and $y$ respectively, and the superscript $k$ represents the iteration index. The generating function describing one iteration is given by

$$\begin{aligned} \varphi_{step}(z,\omega) =& (1-\triangle t(2-\gamma_1)) + \triangle t \frac{1}{2}(1-\gamma_1)(z^{-1} + z + \omega^{-1} + \omega) \\ & + \triangle t \frac{1}{4}\gamma_1(z^{-1}\omega^{-1} + z^{-1}\omega + z\omega^{-1} + z\omega). \end{aligned}$$

We now assume that the scale-space representation at scale $t$ is computed by $n$ iterations with a scale step $\triangle t = \frac{t}{n}$. Then using the fact

$$\varphi_{composed,n}(z,\omega) = \left(\varphi_{step}(z,\omega)\right)^n,$$

the generating function describing the composed transformation is obtained as

$$\varphi_{composed,n}(z,\omega) =$$
$$\left(1+\frac{t}{n}\left(\left(\frac{z^{-1}}{2}-1+\frac{z}{2}\right)+\left(\frac{\omega^{-1}}{2}-1+\frac{\omega}{2}\right)+\gamma_1\left(\frac{z^{-1}}{2}-1+\frac{z}{2}\right)\cdot\left(\frac{\omega^{-1}}{2}-1+\frac{\omega}{2}\right)\right)\right)^n.$$

According to the fact that $\lim_{n\to\infty}(1+\alpha_n/n)^n = e^{\alpha}$ if $\lim_{n\to\infty}\alpha_n = \alpha$,

$$\begin{aligned}\lim_{n\to\infty}\varphi_{composed,n}(z,\omega) =& e^{t\left(\left(\frac{z^{-1}}{2}-1+\frac{z}{2}\right)+\left(\frac{\omega^{-1}}{2}-1+\frac{\omega}{2}\right)+\gamma_1\left(\frac{z^{-1}}{2}-1+\frac{z}{2}\right)\cdot\left(\frac{\omega^{-1}}{2}-1+\frac{\omega}{2}\right)\right)}\\ =& e^{t\left(\frac{z^{-1}}{2}-1+\frac{z}{2}\right)}\cdot e^{t\left(\frac{\omega^{-1}}{2}-1+\frac{\omega}{2}\right)}\cdot e^{\gamma_1 t\left(\frac{z^{-1}}{2}-1+\frac{z}{2}\right)\cdot\left(\frac{\omega^{-1}}{2}-1+\frac{\omega}{2}\right)}\\ =& e^{-t}e^{t(z^{-1}+z)/2}\cdot e^{-t}e^{t(\omega^{-1}+\omega)/2}\cdot e^{\gamma_1 t\left(\frac{z^{-1}}{2}-1+\frac{z}{2}\right)\cdot\left(\frac{\omega^{-1}}{2}-1+\frac{\omega}{2}\right)}\end{aligned} \tag{2.17}$$

follows. $e^{t(z^{-1}+z)/2}$ and $e^{t(\omega^{-1}+\omega)/2}$ are apparently recognized as the generating functions for the modified Bessel functions of integer order. Therefore, $e^{-t}e^{t(z^{-1}+z)/2}$ and $e^{-t}e^{t(\omega^{-1}+\omega)/2}$, respectively, define $T(n;t) = e^{-t}I_n(t)$ and $T(m;t) = e^{-t}I_m(t)$, which are the 1-D discrete analogue of the Gaussian kernel according to Definition 6 (see Section 2.1.2).

As a simplest case ($\gamma_1 = 0$), the expression (2.17) can be expressed in the form of the multiplication of generating functions of the family of the discrete Gaussian kernels

$$\varphi_T(z,\omega) = \sum_{(m,n)\in\mathbb{Z}^2} T(m,n;t) z^m \omega^n,$$

where

$$T(m,n;t) = e^{-t}I_m(t)\cdot e^{-t}I_n(t).$$

However, (2.17) has the term $e^{\gamma_1 t\left(\frac{z^{-1}}{2}-1+\frac{z}{2}\right)\cdot\left(\frac{\omega^{-1}}{2}-1+\frac{\omega}{2}\right)}$ whose value depends on $\gamma_1$, which means that we can not derive a separate discrete kernel when $\gamma_1 \neq 0$, and even worse, $\gamma_1$ is not definite.

On the other hand, the discretization of (2.16) with scale step $\triangle t$ corresponds to the iteration with a kernel given as

$$\begin{pmatrix} \frac{1}{4}\gamma_1\triangle t & \frac{1}{2}(1-\gamma_1)\triangle t & \frac{1}{4}\gamma_1\triangle t \\ \frac{1}{2}(1-\gamma_1)\triangle t & 1-(2-\gamma_1)\triangle t & \frac{1}{2}(1-\gamma_1)\triangle t \\ \frac{1}{4}\gamma_1\triangle t & \frac{1}{2}(1-\gamma_1)\triangle t & \frac{1}{4}\gamma_1\triangle t \end{pmatrix}.$$

This iteration kernel is separable if and only if $\gamma_1 = \triangle t$ (see [12, p. 116-117] for the proof). In that case, the corresponding 1-D kernel is a discrete scale-space kernel if and only if $0 \le \gamma_1 \le \frac{1}{2}$ according to Proposition 7. That is, $\gamma_1$ should not exceed $\frac{1}{2}$.

**Parameter Determination** What remains to be done is to determine the parameters $\gamma_i \in [0,1]$. Although the effect of their value on the scale-space representation could be somehow analyzed, the question about definite parameter determination has been left open in Lindeberg's work.

Let us consider the 2-D case (i.e., the determination of $\gamma_1 \in [0,1]$);

- $\gamma_1 = 0$ :

  The convolution kernel associated with the scale-space representation of the form (2.15) is separable, i.e., $L$ can be given by

  $$L(x,y;t) = \sum_{m=-\infty}^{\infty} T(m;t) \sum_{n=-\infty}^{\infty} T(n;t) f(x-m, y-n) \qquad (t>0),$$

  where $T(n;t) = e^{-t}I_n(t)$. In the separable case the high-dimensional discrete scale-space corresponds to repeated application of the one-dimensional scale-space along each coordinate direction. When we choose this value, however, the point operator $\nabla_5^2$ of the iteration kernel only links the cross points along the $x$- and $y$-axis.

- $\gamma_1 = \frac{1}{3}$ :

  It gives the least rotational asymmetry in the solution to the differential equation (2.15) (see [12, p.117-118] for the proof). The left below kernel is the discrete iteration kernel and the right one is the Laplacian operator when $\triangle t = \gamma_1 = \frac{1}{3}$;

$$\begin{pmatrix} \frac{1}{36} & \frac{1}{9} & \frac{1}{36} \\ \frac{1}{9} & \frac{4}{9} & \frac{1}{9} \\ \frac{1}{36} & \frac{1}{9} & \frac{1}{36} \end{pmatrix} \qquad \begin{pmatrix} \frac{1}{6} & \frac{4}{6} & \frac{1}{6} \\ \frac{4}{6} & -\frac{20}{6} & \frac{4}{6} \\ \frac{1}{6} & \frac{4}{6} & \frac{1}{6} \end{pmatrix}$$

The discrete iteration kernel    The Laplacian operator

- $\gamma_1 = \frac{1}{2}$ :

  This is the boundary value which should not be exceeded for the discrete iteration mentioned in the previous section. The left below kernel corresponds to separated convolution with the one-dimensional binomial kernel $(\frac{1}{4}\ \frac{1}{2}\ \frac{1}{4})$. The right one is the Laplacian operator;

$$\begin{pmatrix} \frac{1}{16} & \frac{1}{8} & \frac{1}{16} \\ \frac{1}{8} & \frac{1}{4} & \frac{1}{8} \\ \frac{1}{16} & \frac{1}{8} & \frac{1}{16} \end{pmatrix} \qquad \begin{pmatrix} \frac{1}{8} & \frac{2}{8} & \frac{1}{8} \\ \frac{2}{8} & -\frac{12}{8} & \frac{2}{8} \\ \frac{1}{8} & \frac{2}{8} & \frac{1}{8} \end{pmatrix}$$

The discrete iteration kernel    The Laplacian operator

- $\gamma_1 = 1$ :

  When we intend to use the iteration kernel, this value is inappropriate. In the case of (2.15), an undesirable situation occurs through differentiation since the cross operator $\nabla^2_{\times^2}$ only links diagonal points.

# 3 An Improved Discrete Scale-Space Formulation for N-D Signals

We reviewed the discrete scale-space theory proposed by Lindeberg in the previous sections. From that review, it becomes clear that convolution of 1-D discrete signals with the 1-D discrete analogue of the Gaussian is equivalent to the analytical solution of the discretized diffusion equation. However, for higher dimensional signals, Lindeberg could not fully derive a definite discrete scale-space formulation, where he posed the open question how to determine the parameter $\gamma \in [0, 1]$ for solving the semi-discretized diffusion equation.

Motivated by this open question, in this section we consider the matter of how to derive a definite scale-space formulation. By solving the problem of determination of parameter $\gamma$, we propose an improved discrete scale-space formulation for higher dimensional signals (concretely, 2-D and 3-D signals) based on a few assumptions.

## 3.1 Preliminaries

### 3.1.1 The Neighborhood Connectivity

It is of importance in dealing with higher dimensional discrete signals how to define the neighborhood connectivity for a given point. Various definitions of the neighborhood connectivity can exist, however, there is not a strict rule to define it. In this work, we assume the neighborhood connectivity as follows:

**Assumption** 9 (The Neighborhood Connectivity)
*"For a given point $p \in \mathbb{Z}^k$, we define its neighborhood ${}^kN(p)$*

$$ {}^kN(p) = \{\xi \in \mathbb{Z}^k : (||p - \xi||_\infty \leq 1) \wedge (\xi \neq p)\}, $$

*for $k \geq 1$."*

According to Assumption 9, the neighborhood connectivity for each dimension is described as follows:

- $k = 1$ (1-D) : Given a point $p = (x)$, its neighborhood ${}^{1}N(p)$ is

$${}^{1}N(p) = \{(x-1), (x+1)\}$$

  with 2 connectivities.

- $k = 2$ (2-D) : Given a point $p = (x, y)$, its neighborhood ${}^{2}N(p)$ is

$$\begin{aligned} {}^{2}N(p) = \{&(x-1, y-1), (x-1, y), (x-1, y+1), (x, y-1), \\ &(x, y+1), (x+1, y-1), (x+1, y), (x+1, y+1)\} \end{aligned}$$

  with 8 connectivities.

- $k = 3$ (3-D) : Given a point $p = (x, y, z)$, its neighborhood ${}^{3}N(p)$ is

$$\begin{aligned} {}^{3}N(p) = \{&(x-1, y-1, z-1), (x-1, y-1, z), (x-1, y-1, z+1), \\ &(x-1, y, z-1), (x-1, y, z), (x-1, y, z+1), \\ &(x-1, y+1, z-1), (x-1, y+1, z), (x-1, y+1, z+1), \\ &(x, y-1, z-1), (x, y-1, z), (x, y-1, z+1), (x, y, z-1), \\ &(x, y, z+1), (x, y+1, z-1), (x, y+1, z), (x, y+1, z+1), \\ &(x+1, y-1, z-1), (x+1, y-1, z), (x+1, y-1, z+1), \\ &(x+1, y, z-1), (x+1, y, z), (x+1, y, z+1), \\ &(x+1, y+1, z-1), (x+1, y+1, z), (x+1, y+1, z+1)\} \end{aligned}$$

  with 26 connectivities.

- $k = N$ (N-D) : Given a point $p = (x_1, x_2, ..., x_N)$, its neighborhood ${}^{N}N(p)$ has $3^N - 1$ connectivities.

Then, the neighborhood connectivity of each dimension can be classified with respect to the distance between a given point and its neighbors:

- 2-D case : Given a point $p = (x, y)$, ${}^2N(p)$ consists of ${}^2N_1(p)$ and ${}^2N_{\sqrt{2}}(p)$ given as

$$\begin{aligned} {}^2N_1(p) &= \{(\xi \in \mathbb{Z}^2) : (\|p - \xi\| = 1) \wedge (\xi \in {}^2N(p))\} \\ &= \{(x, y-1), (x, y+1), (x-1, y), (x+1, y)\} \quad \text{and} \\ {}^2N_{\sqrt{2}}(p) &= \{(\xi \in \mathbb{Z}^2) : (\|p - \xi\| = \sqrt{2}) \wedge (\xi \in {}^2N(p))\} \\ &= \{(x-1, y-1), (x-1, y+1), (x+1, y-1), (x-1, y-1)\}. \end{aligned}$$

- 3-D case : Given a point $p = (x, y, z)$, ${}^3N(p)$ consists of ${}^3N_1(p)$, ${}^3N_{\sqrt{2}}(p)$, and ${}^3N_{\sqrt{3}}(p)$ given as

$$\begin{aligned} {}^3N_1(p) &= \{(\xi \in \mathbb{Z}^3) : (\|p - \xi\| = 1) \wedge (\xi \in {}^3N(p))\} \\ &= \{(x, y, z-1), (x, y, z+1), (x, y-1, z), (x, y+1, z), (x-1, y, z), (x+1, y, z)\}, \\ {}^3N_{\sqrt{2}}(p) &= \{(\xi \in \mathbb{Z}^3) : (\|p - \xi\| = \sqrt{2}) \wedge (\xi \in {}^3N(p))\} \\ &= \{(x-1, y-1, z), (x-1, y+1, z), (x+1, y-1, z), (x+1, y+1, z), \\ &\quad (x-1, y, z-1), (x-1, y, z+1), (x+1, y, z-1), (x+1, y, z+1), \\ &\quad (x, y-1, z-1), (x, y-1, z+1), (x, y+1, z-1), (x, y+1, z+1)\}, \quad \text{and} \\ {}^3N_{\sqrt{3}}(p) &= \{(\xi \in \mathbb{Z}^3) : (\|p - \xi\| = \sqrt{3}) \wedge (\xi \in {}^3N(p))\} \\ &= \{(x-1, y-1, z-1), (x+1, y-1, z-1), (x-1, y+1, z-1), \\ &\quad (x-1, y-1, z+1), (x-1, y+1, z+1), (x+1, y-1, z+1), \\ &\quad (x+1, y+1, z-1), (x+1, y+1, z+1)\}. \end{aligned}$$

- N-D case : Given a point $p = (x_1, x_2, ..., x_N)$, ${}^NN(p)$ consists of ${}^NN_1(p)$, ${}^NN_{\sqrt{2}}(p)$, ..., and ${}^NN_{\sqrt{N}}(p)$. In this case, the number of elements of ${}^NN_{\sqrt{k}}(p)$ is equivalent to the number of the complexions [19] consisting of $k$ elements of which each has 2 complexions that can be formed from $N$ different elements without taking into account their arrangement of order $k$; that is, complexions of $\binom{N}{k}2^k$, $k = 1, 2, ...N$.

[19] In combinatorics, complexions investigate the arrangement or composition of a finite number of elements identified by symbols ([6, p. 773]).

### 3.1.2 The Laplacian of The N-D Discrete Scale-Space Kernel

The Laplacian of the Gaussian is a rotationally symmetric operator ([13]). However, it is not obvious how to define a rotationally symmetric Laplacian of the N-D discrete scale-space kernel. In this section, based on numerical differentiation we define the Laplacian of the N-D discrete scale-space kernel which will be used for describing the discretized diffusion equation in Section 3.2 and in Section 3.3.

On the basis of numerical differentiation (see [2], [6]), functions can be differentiated numerically which is practical when the analytic solution cannot be determined at all. Derivatives are approximated by the difference quotient (difference formula of the first order) such that

$$\frac{\partial f(x)}{\partial x} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \frac{f(x+h) - f(x)}{h} + O(h),$$

where $O(h)$ is the error term of the approximation; the error is linear in $h$. Also, the second derivative is given as

$$\frac{\partial^2 f(x)}{\partial x^2} = \frac{f(x+h) - 2f(x) + f(x-h)}{h^2} + O(h^2),$$

where the error changes quadratically with $h$ ([6, Sec. 12.5.2]). In our work, we assume the step size $h$ to be one and ignore the error term $O(h^2)$, which leads to

$$\frac{\partial^2 f(x)}{\partial x^2} \approx f(x+1) - 2f(x) + f(x-1).$$

Accordingly, the 2-D Laplacian is expressed as

$$\begin{aligned} \nabla^2 f(x,y) &= \frac{\partial^2 f(x,y)}{\partial x^2} + \frac{\partial^2 f(x,y)}{\partial y^2} \\ &\approx f(x,y-1) + f(x,y+1) + f(x-1,y) + f(x+1,y) - 4f(x,y). \end{aligned}$$

For a given discrete higher dimensional signal $f : \mathbb{Z}^N \to \mathbb{R}$ the scale-space representation is generated by the convolution with a one-parameter family of kernels of the form

$$L(x;t) = T(x;t) * f(x) = \sum_{\xi \in \mathbb{Z}^N} T(\xi;t) f(x-\xi),$$

where a family of kernels $T : \mathbb{Z}^N \times \mathbb{R}_+ \rightarrow \mathbb{R}$ satisfies the non-enhancement requirement based on Definition 8; in the case of continuous signals, the Gaussian kernel corresponds to the family of kernels for constructing the scale-space representation. Let us refer to the kernel $T : \mathbb{Z}^N \times \mathbb{R}_+ \rightarrow \mathbb{R}$ as *the N-D discrete scale-space kernel.* The scale-space representation $L$ satisfies

$$\partial_t L(x;t) = A_{ScSp} L(x;t),$$

where $A_{ScSp}$ is an infinitesimal scale-space generator (see for its proof [12, p.109-112]), and this corresponds in 2-D to

$$\begin{aligned}\partial_t L(x,y;t) &= a_1 \nabla^2_{2N_1} L(x,y;t) + a_2 \nabla^2_{2N_{\sqrt{2}}} L(x,y;t) \\ &= a_1 \left(\nabla^2_{2N_1} T(x,y;t) * f(x,y)\right) + a_2 \left(\nabla^2_{2N_{\sqrt{2}}} T(x,y;t) * f(x,y)\right),\end{aligned} \tag{3.18}$$

and in 3-D to

$$\begin{aligned}&\partial_t L(x,y,z;t) \\ &= a_1 \nabla^2_{3N_1} L(x,y,z;t) + a_2 \nabla^2_{3N_{\sqrt{2}}} L(x,y,z;t) + a_3 \nabla^2_{3N_{\sqrt{3}}} L(x,y,z;t) \\ &= a_1 \left(\nabla^2_{3N_1} T(x,y,z;t) * f(x,y,z)\right) + a_2 \left(\nabla^2_{3N_{\sqrt{2}}} T(x,y,z;t) * f(x,y,z)\right) + \\ &\quad a_3 \left(\nabla^2_{3N_{\sqrt{3}}} T(x,y,z;t) * f(x,y,z)\right),\end{aligned} \tag{3.19}$$

for some constants $a_1 \geq 0$, $a_2 \geq 0$, and $a_3 \geq 0$. The constants ($a_1$, $a_2$, and $a_3$) play the role in preserving the rotational symmetry of the Laplacian of the 2-D and 3-D discrete scale-space kernel, and their proper values will be determined in the following sections.

As a consequence, the Laplacian of the N-D discrete scale-space kernel for 2-D and 3-D signals are described as follows:

**Assumption** 10 (The Laplacian of The N-D Discrete Scale-Space Kernel) *"For a family of kernels $T : \mathbb{Z}^N \times \mathbb{R}_+ \rightarrow \mathbb{R}$,*

- *$N = 2$ : 2-D case*

$$\begin{aligned}\nabla^2 T(x,y;t) &= a_1 \nabla^2_{2N_1} T(x,y;t) + a_2 \nabla^2_{2N_{\sqrt{2}}} T(x,y;t), \quad \textit{where} a_1, a_2 \geq 0 \quad \textit{and} \\ \nabla^2_{2N_1} T(x,y;t) &= T(x,y-1;t) + T(x,y+1;t) + T(x-1,y;t) + T(x+1,y;t) - 4T(x,y;t), \\ \nabla^2_{2N_{\sqrt{2}}} T(x,y;t) &= T(x-1,y-1;t) + T(x-1,y+1;t) + T(x+1,y-1;t) + \\ &\quad T(x-1,y-1;t) - 4T(x,y;t).\end{aligned}$$

- $N = 3$ : *3-D case*

$$
\begin{aligned}
\nabla^2 T(x,y,z;t) &= a_1 \nabla^2_{{}^3N_1} T(x,y,z;t) + a_2 \nabla^2_{{}^3N_{\sqrt{2}}} T(x,y,z;t) + a_3 \nabla^2_{{}^3N_{\sqrt{3}}} T(x,y,z;t), \\
\textit{where} \quad & a_1, a_2, a_3 \geq 0 \quad \textit{and} \\
\nabla^2_{{}^3N_1} T(x,y,z;t) &= T(x,y,z-1;t) + T(x,y,z+1;t) + T(x,y-1,z;t) + \\
& T(x,y+1,z;t) + T(x-1,y,z;t) + T(x+1,y,z;t) - 6T(x,y,z;t), \\
\nabla^2_{{}^3N_{\sqrt{2}}} T(x,y,z;t) &= T(x-1,y-1,z;t) + T(x-1,y+1,z;t) + T(x+1,y-1,z;t) + \\
& T(x+1,y+1,z;t) + T(x-1,y,z-1;t) + T(x-1,y,z+1;t) + \\
& T(x+1,y,z-1;t) + T(x-1,y,z-1;t) + T(x,y-1,z-1;t) + \\
& T(x,y-1,z+1;t) + T(x,y+1,z-1;t) + T(x,y+1,z+1;t) - \\
& 12T(x,y,z;t), \\
\nabla^2_{{}^3N_{\sqrt{3}}} T(x,y,z;t) &= T(x-1,y-1,z-1;t) + T(x+1,y-1,z-1;t) + \\
& T(x-1,y+1,z-1;t) + T(x-1,y-1,z+1;t) + \\
& T(x-1,y+1,z+1;t) + T(x+1,y-1,z+1;t) + \\
& T(x+1,y+1,z-1;t) + T(x+1,y+1,z+1;t) - 8T(x,y,z;t)."
\end{aligned}
$$

The Laplacian of the 2-D and 3-D discrete scale-space kernels defined in Assumption 10 are similar to those defined by Lindeberg [12, p.105] in that the two directions (i.e., ${}^2N_1$ and ${}^2N_{\sqrt{2}}$) in 2-D and three directions (i.e., ${}^3N_1$, ${}^3N_{\sqrt{2}}$, and ${}^3N_{\sqrt{3}}$) in 3-D are considered for the Laplacian of the discrete scale-space kernel. However, with respect to the coefficients of the Laplacian of ${}^2N_{\sqrt{2}}$ as well as of ${}^3N_{\sqrt{2}}$ and of ${}^3N_{\sqrt{3}}$ we did not fix any ambiguous coefficients, which is obviously different from the definition of Lindeberg; Lindeberg set the coefficient of the Laplacian of ${}^2N_{\sqrt{2}}$ to $\frac{1}{2}$, and that of ${}^3N_{\sqrt{2}}$ as well as of ${}^3N_{\sqrt{3}}$ to $\frac{1}{4}$ (see the footnotes on page 20 and 21).

## 3.2 The 2-D Discrete Scale-Space Formulation

The equation (3.18) can be expressed as a normalized form

$$\partial_t L(x,y;t) = \frac{1}{2}\nabla^2 L(x,y;t) = \frac{1}{2}\left((1-\gamma)\nabla^2_{{}^2N_1} L(x,y;t) + \gamma\nabla^2_{{}^2N_{\sqrt{2}}} L(x,y;t)\right)$$

for $\gamma \in [0,1]$ : If $\gamma = 0$, then only the points linked by the neighborhood connectivity ${}^2N_1$ are considered, and to the contrary if $\gamma = 1$ then only the points linked by the neighborhood connectivity ${}^2N_{\sqrt{2}}$ are considered. This equation can be further discretized using Euler's explicit method with the scale step $\triangle t$;

$$\begin{aligned}
L^{k+1}_{x,y} =& L^k_{x,y} + \triangle t\left(\partial_t L^k_{x,y}\right) \\
=& L^k_{x,y} + \triangle t\frac{1}{2}\left((1-\gamma)\nabla^2_{{}^2N_1} L + \gamma\nabla^2_{{}^2N_{\sqrt{2}}} L\right) \\
=& L^k_{x,y} + \triangle t\frac{1}{2}(1-\gamma)\left(L^k_{x-1,y} + L^k_{x+1,y} + L^k_{x,y-1} + L^k_{x,y+1} - 4L^k_{x,y}\right) \\
&+ \triangle t\frac{1}{2}\gamma\left(L^k_{x-1,y-1} + L^k_{x-1,y+1} + L^k_{x+1,y-1} + L^k_{x+1,y+1} - 4L^k_{x,y}\right) \\
=& (1-2\triangle t)L^k_{x,y} + \frac{1}{2}\triangle t(1-\gamma)\left(L^k_{x-1,y} + L^k_{x+1,y} + L^k_{x,y-1} + L^k_{x,y+1}\right) \\
&+ \frac{1}{2}\triangle t\gamma\left(L^k_{x-1,y-1} + L^k_{x-1,y+1} + L^k_{x+1,y-1} + L^k_{x+1,y+1}\right),
\end{aligned} \tag{3.20}$$

where the subscript $x$ and $y$ denote the spatial coordinates, and the superscript $k$ represents the iteration index.

The discretization of (3.20) with the scale step $\triangle t$ corresponds to the iteration with a kernel given as

$$L^{k+1} = T_{\triangle t} * L^k,$$

where $L^0 = f(x,y)$ and the 2-D iteration kernel is

$$T_{\triangle t} = \begin{pmatrix} \frac{1}{2}\gamma\triangle t & \frac{1}{2}(1-\gamma)\triangle t & \frac{1}{2}\gamma\triangle t \\ \frac{1}{2}(1-\gamma)\triangle t & 1-2\triangle t & \frac{1}{2}(1-\gamma)\triangle t \\ \frac{1}{2}\gamma\triangle t & \frac{1}{2}(1-\gamma)\triangle t & \frac{1}{2}\gamma\triangle t \end{pmatrix}. \tag{3.21}$$

### 3.2.1 Parameter Determination in 2-D

The iteration kernel of the form (3.21) describing one iteration with step scale $\triangle t$ is derived from discretization of the diffusion equation of (3.20), where parameter $\gamma$ plays the role in preserving the rotational symmetry of the 2-D discrete scale-space kernel. For a given 2-D signal $f : \mathbb{Z}^2 \to \mathbb{R}$, the scale-space representation is generated by the convolution with the 2-D discrete scale-space kernel $T$ constructed by $n$ iterations given a certain scale $t$ ($t = n \cdot \triangle t$) such as

$$\begin{aligned} L(x, y; t) &= T(x, y; t) * f(x, y) \\ &= \underbrace{T_{\triangle t} * \cdots * (T_{\triangle t}}_{n} * f(x, y)). \end{aligned}$$

Here, on one hand, it is important to recapitulate that rotational symmetry related to spatial isotropy is one of the basic principles of linear scale-space, which requires the discrete scale-space kernel $T$ to be rotationally symmetric. On the other hand, however, it is noticeable that rotational symmetry (or rotational invariance) is not a primary factor to be aimed at in the discrete case since one is locked to a fixed square grid, and furthermore it is also far from obvious as to what is meant by spatial isotropy on a discrete grid ([12, p.117]). Instead of finding spatial isotropy or rotational symmetry, therefore, it would be more reasonable to point out the lack of spatial anisotropy or rotational asymmetry. In order to measure the least possible rotational asymmetry, let us use the Fourier transform of the generating function of the iteration kernel.

Using a well-known relationship between the Fourier transform and the generating function (see [14, Chap. 3]), the generating function describing one iteration of (3.20) is

$$\begin{aligned} {}^{2}\varphi_{step}(z, \chi) =& (1 - 2\triangle t) + \frac{1}{2}\triangle t(1-\gamma)\left(z^{-1} + z + \chi^{-1} + \chi\right) + \\ & \frac{1}{2}\triangle t\gamma\left(z^{-1}\chi^{-1} + z^{-1}\chi + z\chi^{-1} + z\chi\right), \end{aligned}$$

and we obtain the generating function describing the composed transformation ($\triangle t = \frac{t}{n}$);

$$\begin{gathered} {}^{2}\varphi_{composed,n}(z, \chi) = \left({}^{2}\varphi_{step}(z, \chi)\right)^{n} = \\ \left(1 + \frac{t}{n}\left(-2 + (1-\gamma)\left(\frac{z^{-1} + z + \chi^{-1} + \chi}{2}\right) + \gamma\left(\frac{z^{-1}\chi^{-1} + z^{-1}\chi + z\chi^{-1} + z\chi}{2}\right)\right)\right)^{n}. \end{gathered}$$

Based on the fact that $\lim_{n\to\infty}(1 + \alpha_n/n)^n = e^{\alpha}$ if $\lim_{n\to\infty}\alpha_n = \alpha$, the generating function of the kernel describing the transformation from the original signal to the representation at a certain scale $t$ is given by

$$\begin{aligned}{}^2\varphi_T(z,\chi) &= \sum_{(m,n)\in\mathbb{Z}^2} T(m,n;t) z^m \chi^n \\ &= e^{t\left(-2+(1-\gamma)\left(\frac{z^{-1}+z+\chi^{-1}+\chi}{2}\right)+\gamma\left(\frac{z^{-1}\chi^{-1}+z^{-1}\chi+z\chi^{-1}+z\chi}{2}\right)\right)},\end{aligned}$$

and its Fourier transform is derived by replacing the complex variables $z$ and $\chi$ with the complex variables $e^{-iu}$ and $e^{-iv}$ as

$$\mathcal{F}\left({}^2\varphi_T(z,\chi)\right) = {}^2\psi_T(e^{-iu}, e^{-iv}),$$

which is expressed using Euler's formula[20] as

$$(3.22) \qquad {}^2\psi_T(\cos u - i\sin u, \cos v - i\sin v) = e^{t(-2+(1-\gamma)(\cos u+\cos v)+\gamma 2\cos u\cos v)}.$$

Then we transform (3.22) into polar coordinates given a fixed value of the radius $r$ and an angular variable $\phi$ such that $u = r\cos\phi$ and $v = r\sin\phi$:

$${}^2\psi_T(r,\phi) = e^{(t\cdot k(r,\phi))},$$

where

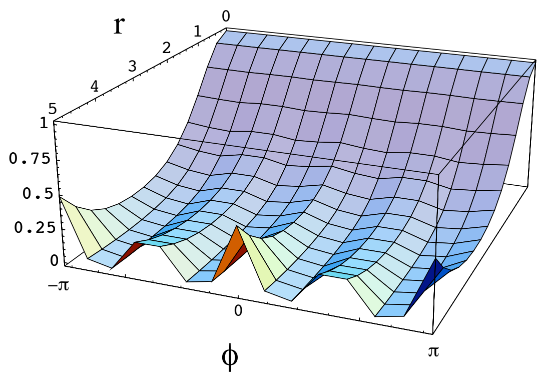
$$\begin{aligned}k(r,\phi) = -2 &+ (1-\gamma)(\cos(r\cos\phi) + \cos(r\sin\phi)) \\ &+ \gamma 2\cos(r\cos\phi)\cos(r\sin\phi).\end{aligned}$$

Fig. 1-(a) and 1-(b) depict ${}^2\psi_T(r,\phi)$ when $\gamma = 0$ and $\gamma = 1$. We can easily recognize from Fig. 1-(a) that four obvious angular variations appear at $|\phi| = 0, \frac{\pi}{2}, \pi$, which reflects the four-point-link of the neighborhood connectivity ${}^2N_1$. Also there are salient angular variations at $|\phi| = \frac{\pi}{4}, \frac{3\pi}{4}$ from Fig. 1-(b), which can be interpreted as a reflection of the diagonal four-point-link of the neighborhood connectivity ${}^2N_{\sqrt{2}}$.

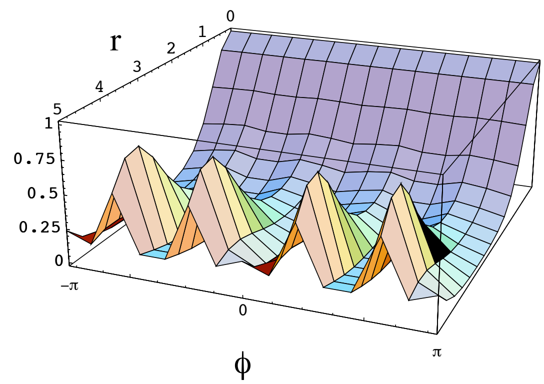
$k(r,\phi)$ consists of three variables, i.e., $r$, $\phi$, and $\gamma$, and we intend to determine the value of $\gamma$ which gives the smallest angular variation of $\phi$ for a fixed value $r$. For examining the

[20]Exponential function with imaginary argument ([6, p. 278]) :

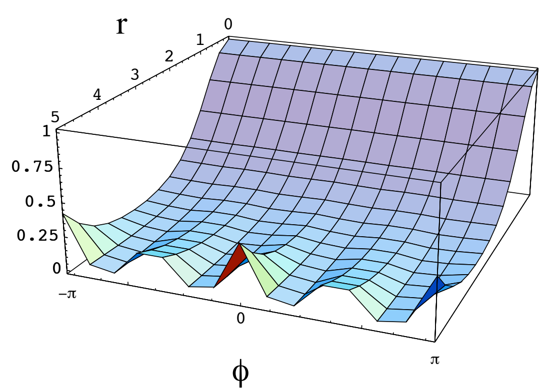
$$e^{ix} = \cos x + i\sin x.$$




(a) $\gamma = 0$




(b) $\gamma = 1$




(c) $\gamma = \frac{1}{5}$

Figure 1: ${}^2\psi_T(r, \phi)$ (for convenience, we here fix scale parameter $t$ to one)

$\phi$-dependency of $\gamma$ from $k(r,\phi)$, we expand the MacLaurin series of $k(r,\phi)$ with respect to $r$ with the help of Mathematica ([16]);

$$\begin{aligned} k(r\cos\phi, r\sin\phi) &= O(r^2) = -\frac{1+\gamma}{2}r^2 + O(r^3) = -\frac{1+\gamma}{2}r^2 + O(r^4) \\ &= -\frac{1+\gamma}{2}r^2 + \frac{3+9\gamma+(1-5\gamma)\cos 4\phi}{96}r^4 + O(r^5) \\ &= -\frac{1+\gamma}{2}r^2 + \frac{3+9\gamma+(1-5\gamma)\cos 4\phi}{96}r^4 + O(r^6). \end{aligned}$$

It is clear from the MacLaurin series above that if $\gamma = \frac{1}{5}$ then the $\phi$-dependency decreases as $r^6$ instead of as $r^4$. Fig. 1-(c) illustrates ${}^2\psi_T(r,\phi)$ when $\gamma = \frac{1}{5}$, which shows that angular variation is distributed into the whole range of $\phi$ (i.e., $-\pi \leq \phi \leq \pi$). Therefore, it can be said that the value $\frac{1}{5}$ of $\gamma$ gives the smallest angular variation, which implies that $\gamma = \frac{1}{5}$ is the parameter value that brings the least possible rotational asymmetry into the 2-D discrete scale-space kernel.

Consequently, the corresponding 2-D iteration kernel by replacing the determined $\gamma$ value as $\frac{1}{5}$ that gives the least rotational asymmetry to the 2-D discrete scale-space kernel is

$$T_{\triangle t} = \begin{pmatrix} \frac{1}{10}\triangle t & \frac{2}{5}\triangle t & \frac{1}{10}\triangle t \\ \frac{2}{5}\triangle t & 1-2\triangle t & \frac{2}{5}\triangle t \\ \frac{1}{10}\triangle t & \frac{2}{5}\triangle t & \frac{1}{10}\triangle t \end{pmatrix}, \tag{3.23}$$

where $\triangle t > 0$.

### 3.2.2 Separable Iteration Kernel in 2-D

Considering economy of computation when iteration increases, it would be reasonable the iteration kernel to be separable; the higher the dimension of computation is, the more efficient separable filters are ([8], [13]).

The iteration kernel of the form (3.23) is symmetric and the sum of its coefficients is one. Therefore, if and only if the kernel can be written as a kernel

$$\begin{pmatrix} a & 1-2a & a \end{pmatrix}$$

convolved with itself for $a \geq 0$, then it can be said that the 2-D kernel is separable (see [12, p. 116-117]). In this case, the kernel $(a, 1-2a, a)$ is a discrete kernel if and only if

$$(1-2a)^2 \geq 4 \cdot a \cdot a,$$

which leads to the condition $0 < a \leq \frac{1}{4}$ (see for its proof [11, p. 238]):

**Assumption** 11 (2-D Separable Iteration Kernel)

*"Let us assume that the 2-D iteration kernel describing the discretized diffusion equation is constructed by convolution of the 1-D kernel with itself as*

$$\begin{pmatrix} a & 1-2a & a \end{pmatrix} * \begin{pmatrix} a \\ 1-2a \\ a \end{pmatrix} = \begin{pmatrix} \frac{1}{10}\triangle t & \frac{2}{5}\triangle t & \frac{1}{10}\triangle t \\ \frac{2}{5}\triangle t & 1-2\triangle t & \frac{2}{5}\triangle t \\ \frac{1}{10}\triangle t & \frac{2}{5}\triangle t & \frac{1}{10}\triangle t \end{pmatrix}$$

*for* $0 < a \leq \frac{1}{4}$.*"*

From Assumption 11,

$$\begin{pmatrix} a^2 & a(1-2a) & a^2 \\ a(1-2a) & (1-2a)^2 & a(1-2a) \\ a^2 & a(1-2a) & a^2 \end{pmatrix} = \begin{pmatrix} \frac{1}{10}\triangle t & \frac{2}{5}\triangle t & \frac{1}{10}\triangle t \\ \frac{2}{5}\triangle t & 1-2\triangle t & \frac{2}{5}\triangle t \\ \frac{1}{10}\triangle t & \frac{2}{5}\triangle t & \frac{1}{10}\triangle t \end{pmatrix}$$

we obtain

$$a = \frac{1}{6} \qquad \text{and} \qquad \triangle t = \frac{5}{18},$$

which means the 2-D iteration kernel is separable into the 1-D discrete scale-space kernel $\left(\frac{1}{6}\frac{2}{3}\frac{1}{6}\right)$ when $\triangle t = \frac{5}{18}$.

Consequently, the 2-D separable iteration kernel and the Laplacian of the 2-D discrete scale-space kernel are given by

(3.24)

$$\begin{pmatrix} \frac{1}{36} & \frac{1}{9} & \frac{1}{36} \\ \frac{1}{9} & \frac{4}{9} & \frac{1}{9} \\ \frac{1}{36} & \frac{1}{9} & \frac{1}{36} \end{pmatrix} \qquad \begin{pmatrix} \frac{1}{5} & \frac{4}{5} & \frac{1}{5} \\ \frac{4}{5} & -\frac{20}{5} & \frac{4}{5} \\ \frac{1}{5} & \frac{4}{5} & \frac{1}{5} \end{pmatrix} .$$

The 2-D iteration kernel — The Laplacian of the 2-D discrete scale-space kernel

## 3.3 The 3-D Discrete Scale-Space Formulation

Since the discrete scale-space formulation for 3-D signals is analogous to that for 2-D signals, we will not recapitulate the details shown in the previous sections.

For a given discrete signal $f : \mathbb{Z}^3 \to \mathbb{R}$, (3.19) can be expressed as a normalized form

$$\partial_t L = \frac{1}{2}\Big((1-\gamma_1-\gamma_2)\nabla^2_{3N_1} L + \gamma_1 \nabla^2_{3N_{\sqrt{2}}} L + \gamma_2 \nabla^2_{3N_{\sqrt{3}}} L\Big)$$

for $\gamma_1, \gamma_2 \in [0,1]$, which can be further discretized using Euler's explicit method with scale step $\triangle t$;

(3.25)

$$\begin{aligned}
L^{k+1}_{x,y,z} =& L^k_{x,y,z} + \triangle t\left(\partial_t L^k_{x,y,z}\right)\\
=& L^k_{x,y,z} + \triangle t \frac{1}{2}\Big((1-\gamma_1-\gamma_2)\nabla^2_{3N_1} L + \gamma_1 \nabla^2_{3N_{\sqrt{2}}} L + \gamma_2 \nabla^2_{3N_{\sqrt{3}}} L\Big)\\
=& L^k_{x,y,z} + \triangle t\Bigg(\frac{1}{2}(1-\gamma_1-\gamma_2)(L^k_{x,y,z-1} + L^k_{x,y,z+1} + L^k_{x,y-1,z} + L^k_{x,y+1,z} + L^k_{x-1,y,z} +\\
& L^k_{x+1,y,z} - 6L^k_{x,y,z}) + \frac{1}{2}\gamma_1(L^k_{x-1,y-1,z} + L^k_{x-1,y+1,z} + L^k_{x+1,y-1,z} + L^k_{x+1,y+1,z} +\\
& L^k_{x-1,y,z-1} + L^k_{x-1,y,z+1} + L^k_{x+1,y,z-1} + L^k_{x-1,y,z-1} + L^k_{x,y-1,z-1} + L^k_{x,y-1,z+1} +\\
& L^k_{x,y+1,z-1} + L^k_{x,y+1,z+1} - 12L^k_{x,y,z}) + \frac{1}{2}\gamma_2(L^k_{x-1,y-1,z-1} + L^k_{x+1,y-1,z-1} + L^k_{x-1,y+1,z-1} +\\
& L^k_{x-1,y-1,z+1} + L^k_{x-1,y+1,z+1} + L^k_{x+1,y-1,z+1} + L^k_{x+1,y+1,z-1} + L^k_{x+1,y+1,z+1} - 8L^k_{x,y,z})\Bigg)\\
=& (1-\triangle t(3+3\gamma_1+\gamma_2))L^k_{x,y,z} +\\
& \frac{1}{2}\triangle t(1-\gamma_1-\gamma_2)(L^k_{x,y,z-1} + L^k_{x,y,z+1} + L^k_{x,y-1,z} + L^k_{x,y+1,z} + L^k_{x-1,y,z} + L^k_{x+1,y,z}) +\\
& \frac{1}{2}\triangle t\gamma_1(L^k_{x-1,y-1,z} + L^k_{x-1,y+1,z} + L^k_{x+1,y-1,z} + L^k_{x+1,y+1,z} + L^k_{x-1,y,z-1} + L^k_{x-1,y,z+1} +\\
& L^k_{x+1,y,z-1} + L^k_{x-1,y,z-1} + L^k_{x,y-1,z-1} + L^k_{x,y-1,z+1} + L^k_{x,y+1,z-1} + L^k_{x,y+1,z+1}) +\\
& \frac{1}{2}\triangle t\gamma_2(L^k_{x-1,y-1,z-1} + L^k_{x+1,y-1,z-1} + L^k_{x-1,y+1,z-1} + L^k_{x-1,y-1,z+1} + L^k_{x-1,y+1,z+1} +\\
& L^k_{x+1,y-1,z+1} + L^k_{x+1,y+1,z-1} + L^k_{x+1,y+1,z+1}),
\end{aligned}$$

where the subscript $x$, $y$, and $z$ denote the spatial coordinates, and the superscript $k$ represents the iteration index.

The discretization of (3.25) with the scale step $\triangle t$ corresponds to the iteration with a kernel given as

$$L^{k+1} = T_{\triangle t} * L^k,$$

where $L^0 = f(x, y, z)$ and the iteration kernel $T_{\triangle t}$ is

$$
\begin{aligned}
z-1: &\begin{pmatrix} \frac{1}{2}\triangle t\gamma_2 & \frac{1}{2}\triangle t\gamma_1 & \frac{1}{2}\triangle t\gamma_2 \\ \frac{1}{2}\triangle t\gamma_1 & \frac{1}{2}\triangle t(1-\gamma_1-\gamma_2) & \frac{1}{2}\triangle t\gamma_1 \\ \frac{1}{2}\triangle t\gamma_2 & \frac{1}{2}\triangle t\gamma_1 & \frac{1}{2}\triangle t\gamma_2 \end{pmatrix}, \\
z: &\begin{pmatrix} \frac{1}{2}\triangle t\gamma_1 & \frac{1}{2}\triangle t(1-\gamma_1-\gamma_2) & \frac{1}{2}\triangle t\gamma_1 \\ \frac{1}{2}\triangle t(1-\gamma_1-\gamma_2) & 1-\triangle t(3+3\gamma_1+\gamma_2) & \frac{1}{2}\triangle t(1-\gamma_1-\gamma_2) \\ \frac{1}{2}\triangle t\gamma_1 & \frac{1}{2}\triangle t(1-\gamma_1-\gamma_2) & \frac{1}{2}\triangle t\gamma_1 \end{pmatrix}, \\
z+1: &\begin{pmatrix} \frac{1}{2}\triangle t\gamma_2 & \frac{1}{2}\triangle t\gamma_1 & \frac{1}{2}\triangle t\gamma_2 \\ \frac{1}{2}\triangle t\gamma_1 & \frac{1}{2}\triangle t(1-\gamma_1-\gamma_2) & \frac{1}{2}\triangle t\gamma_1 \\ \frac{1}{2}\triangle t\gamma_2 & \frac{1}{2}\triangle t\gamma_1 & \frac{1}{2}\triangle t\gamma_2 \end{pmatrix},
\end{aligned}
\tag{3.26}
$$

which is described with respect to $z-1$, $z$, and $z+1$, respectively, along the $z$-axis.

### 3.3.1 Parameter Determination in 3-D

In the same way as done for the 2-D case, we here impose the restriction of the least possible rotational asymmetry on the 3-D discrete scale-space kernel $T$. The parameter determination of $\gamma_1$ and $\gamma_2$ which give a least possible rotational asymmetry in the 3-D discrete scale-space kernel can be derived in the same way as done in 2-D case.

The generating function describing one iteration of (3.25) is

$$
\begin{aligned}
{}^3\varphi_{step}(z,\chi,\tau) \;=\; & (1-\triangle t(3+3\gamma_1+\gamma_2)) + \frac{1}{2}\triangle t(1-\gamma_1-\gamma_2)(z^{-1}+z+\chi^{-1}+\chi+ \\
& \tau^{-1}+\tau) + \frac{1}{2}\triangle t\gamma_1(z^{-1}\chi^{-1}+z^{-1}\chi+z\chi^{-1}+z\chi+z^{-1}\tau^{-1}+z^{-1}\tau+ \\
& z\tau^{-1}+z\tau+\chi^{-1}\tau^{-1}+\chi^{-1}\tau+\chi\tau^{-1}+\chi\tau) + \frac{1}{2}\triangle t\gamma_2(z^{-1}\chi^{-1}\tau^{-1}+ \\
& z\chi^{-1}\tau^{-1}+z^{-1}\chi\tau^{-1}+z^{-1}\chi^{-1}\tau+z^{-1}\chi\tau+z\chi^{-1}\tau+z\chi\tau^{-1}+z\chi\tau),
\end{aligned}
$$

and we obtain the generating function describing the composed transformation ($\triangle t = \frac{t}{n}$);

$$
\begin{aligned}
&{}^{3}\varphi_{composed,n}(z,\chi,\tau) = \left({}^{3}\varphi_{step}(z,\chi,\tau)\right)^n = \\
&\left(1+\frac{t}{n}\left(-(3+3\gamma_1+\gamma_2)+(1-\gamma_1-\gamma_2)\left(\frac{z^{-1}+z+\chi^{-1}+\chi+\tau^{-1}+\tau}{2}\right)+\right.\right.\\
&\gamma_1\left(\frac{z^{-1}\chi^{-1}+z^{-1}\chi+z\chi^{-1}+z\chi+z^{-1}\tau^{-1}+z^{-1}\tau+z\tau^{-1}+z\tau+\chi^{-1}\tau^{-1}+\chi^{-1}\tau+\chi\tau^{-1}+\chi\tau}{2}\right)\\
&\left.\left.+\gamma_2\left(\frac{z^{-1}\chi^{-1}\tau^{-1}+z\chi^{-1}\tau^{-1}+z^{-1}\chi\tau^{-1}+z^{-1}\chi^{-1}\tau+z^{-1}\chi\tau+z\chi^{-1}\tau+z\chi\tau^{-1}+z\chi\tau}{2}\right)\right)\right)^n.
\end{aligned}
$$

Based on the fact that $\lim_{n\to\infty}(1+\alpha_n/n)^n = e^{\alpha}$ if $\lim_{n\to\infty}\alpha_n = \alpha$, the generating function of the kernel describing the transformation from the original signal to the representation at a certain scale $t$ is given by

$$
\begin{aligned}
&{}^{3}\varphi_T(z,\chi,\tau) = \sum_{(m,n,l)\in\mathbb{Z}^3} T(m,n,l;t)z^m\chi^n\tau^l = \\
&\exp\Big(t\Big(-(3+3\gamma_1+\gamma_2)+(1-\gamma_1-\gamma_2)(z^{-1}+z+\chi^{-1}+\chi+\tau^{-1}+\tau)/2+\\
&\quad\gamma_1(z^{-1}\chi^{-1}+z^{-1}\chi+z\chi^{-1}+z\chi+z^{-1}\tau^{-1}+z^{-1}\tau+z\tau^{-1}+z\tau+\chi^{-1}\tau^{-1}+\\
&\quad\chi^{-1}\tau+\chi\tau^{-1}+\chi\tau)/2+\gamma_2(z^{-1}\chi^{-1}\tau^{-1}+z\chi^{-1}\tau^{-1}+z^{-1}\chi\tau^{-1}+z^{-1}\chi^{-1}\tau+\\
&\quad z^{-1}\chi\tau+z\chi^{-1}\tau+z\chi\tau^{-1}+z\chi\tau)/2\Big)\Big),
\end{aligned}
$$

and its Fourier transform is

(3.27)

$$
\begin{aligned}
&\mathcal{F}\left({}^{3}\varphi_T(z,\chi,\tau)\right) = {}^{3}\psi_T(\underbrace{e^{-iu}}_{\cos u - i\sin u},\underbrace{e^{-iv}}_{\cos v - i\sin v},\underbrace{e^{-i\omega}}_{\cos\omega - i\sin\omega}) = \\
&e^{t(-(3+3\gamma_1+\gamma_2)+(1-\gamma_1+\gamma_2)(\cos u+\cos v+\cos\omega)+\gamma_1 2(\cos u\cos v+\cos u\cos\omega+\cos v\cos\omega)+\gamma_2 4\cos u\cos v\cos\omega)}.
\end{aligned}
$$

Then we transform (3.27) into a function of the polar angles $\phi$ and $\theta$ given a fixed value of the radius $r$ such that $u = r\cos\phi\sin\theta$, $v = r\sin\phi\sin\theta$, and $\omega = r\cos\theta$:

$$
{}^{3}\psi_T(r,\phi,\theta) = e^{(t\cdot k(r,\phi,\theta))},
$$

where

$$\begin{aligned} k(r,\phi,\theta) &= -(3+3\gamma_1+\gamma_2)+ \\ &\quad (1-\gamma_1-\gamma_2)(\cos(r\cos\phi\sin\theta)+\cos(r\sin\phi\sin\theta)+\cos(r\cos\theta))+ \\ &\quad 2\gamma_1(\cos(r\cos\phi\sin\theta)\cos(r\sin\phi\sin\theta)+\cos(r\cos\phi\sin\theta)\cos(r\cos\theta)+ \\ &\qquad \cos(r\sin\phi\sin\theta)\cos(r\cos\theta))+ \\ &\quad 4\gamma_2\cos(r\cos\phi\sin\theta)\cos(r\sin\phi\sin\theta)\cos(r\cos\theta). \end{aligned}$$

Now we want to find the values of $\gamma_1$ and $\gamma_2$ which give the smallest angular variation of $\phi$ and $\theta$ for a fixed value of $r$. For this, we examine the $\phi$- and $\theta$-dependency of $\gamma_1$ and $\gamma_2$ from $k(r,\phi,\theta)$ above. In the same way as done in 2-D, we expand the MacLaurin series of $k(r,\phi,\theta)$ with respect to $r$ with the help of Mathematica ([16]);

$$\begin{aligned} &k(r,\phi,\theta) = \\ &\frac{-(1+3\gamma_1+3\gamma_2)}{2}r^2+\frac{1}{24}\Bigg((1+3\gamma_1+3\gamma_2)\cos^4\theta+12(\gamma_1+2\gamma_2)\cos^2\theta\sin^2\theta- \\ &\frac{(-3(1+5\gamma_1+7\gamma_2)+(-1+3\gamma_1+9\gamma_2)\cos(4\phi))\sin^4\theta}{4}\Bigg)r^4+O(r^6). \end{aligned} \tag{3.28}$$

If the term $-1+3\gamma_1+9\gamma_2$ is null ($\gamma_1,\ \gamma_2>0$) in (3.28), i.e, $\gamma_1=\frac{1-9\gamma_2}{3}$ ($0<\gamma_1<\frac{1}{3}, 0<\gamma_2<\frac{1}{9}$), the $\phi$-dependency of $k(r,\phi,\theta)$ decreases as $r^6$ instead of as $r^4$. When $-1+3\gamma_1+9\gamma_2=0$, (3.28) is rewritten as

$$\begin{aligned} k(r,\phi,\theta) &= -\frac{2-6\gamma_2}{2}r^2+\frac{1}{24}(2-6\gamma_2)\underbrace{\left(\cos^4\theta+\cos^2\theta\sin^2\theta+\sin^4\theta\right)}_{1}r^4+O(r^6) \\ &= -(1-3\gamma_2)r^2+\frac{1}{2}(1-3\gamma_2)r^4+O(r^6). \end{aligned}$$

It is clear from the derivation above that if $-1+3\gamma_1+9\gamma_2=0$ ($0<\gamma_1<\frac{1}{3}, 0<\gamma_2<\frac{1}{9}$), then the $\theta$-dependency also decreases as $r^6$ instead of as $r^4$.

### 3.3.2 Separable Iteration Kernel in 3-D

Since the 3-D iteration kernel of the form (3.26) is symmetric and the sum of its coefficients is one, if and only if the kernel can be written as a kernel

$$\begin{pmatrix} a & 1-2a & a \end{pmatrix}$$

convolved with itself

$$\begin{pmatrix} a & 1-2a & a \end{pmatrix}_x * \begin{pmatrix} a & 1-2a & a \end{pmatrix}_y * \begin{pmatrix} a & 1-2a & a \end{pmatrix}_z =$$

$$\begin{pmatrix} a^2 & a(1-2a) & a^2 \\ a(1-2a) & (1-2a)^2 & a(1-2a) \\ a^2 & a(1-2a) & a^2 \end{pmatrix} * \begin{pmatrix} a & 1-2a & a \end{pmatrix}_z =$$

$$\underbrace{\begin{pmatrix} a^2(1-2a) & a(1-2a)^2 & a^2(1-2a) \\ a(1-2a)^2 & (1-2a)^3 & a(1-2a)^2 \\ a^2(1-2a) & a(1-2a)^2 & a^2(1-2a) \end{pmatrix}}_{z}, \quad \underbrace{\begin{pmatrix} a^3 & a^2(1-2a) & a^3 \\ a^2(1-2a) & a(1-2a)^2 & a^2(1-2a) \\ a^3 & a^2(1-2a) & a^3 \end{pmatrix}}_{z\pm1} .$$

for $0 < a \leq \frac{1}{4}$, then it can be said that the 3-D iteration kernel is separable:

**Assumption** 12 (3-D Separable Iteration Kernel)

*"Let us assume that the 3-D iteration kernel of the form (3.26) describing the discretized diffusion equation is constructed by convolution of the 1-D kernel with itself as*

$$\begin{pmatrix} a & 1-2a & a \end{pmatrix}_x * \begin{pmatrix} a & 1-2a & a \end{pmatrix}_y * \begin{pmatrix} a & 1-2a & a \end{pmatrix}_z$$

*for $0 < a \leq \frac{1}{4}$."*

From Assumption 12,

$$z \pm 1 : \begin{pmatrix} a^3 & a^2(1-2a) & a^3 \\ a^2(1-2a) & a(1-2a)^2 & a^2(1-2a) \\ a^3 & a^2(1-2a) & a^3 \end{pmatrix} = \begin{pmatrix} \frac{1}{2}\triangle t\gamma_2 & \frac{1}{2}\triangle t\gamma_1 & \frac{1}{2}\triangle t\gamma_2 \\ \frac{1}{2}\triangle t\gamma_1 & \frac{1}{2}\triangle t(1-\gamma_1-\gamma_2) & \frac{1}{2}\triangle t\gamma_1 \\ \frac{1}{2}\triangle t\gamma_2 & \frac{1}{2}\triangle t\gamma_1 & \frac{1}{2}\triangle t\gamma_2 \end{pmatrix},$$

$$z : \begin{pmatrix} a^2(1-2a) & a(1-2a)^2 & a^2(1-2a) \\ a(1-2a)^2 & (1-2a)^3 & a(1-2a)^2 \\ a^2(1-2a) & a(1-2a)^2 & a^2(1-2a) \end{pmatrix} =$$

$$\begin{pmatrix} \frac{1}{2}\triangle t\gamma_1 & \frac{1}{2}\triangle t(1-\gamma_1-\gamma_2) & \frac{1}{2}\triangle t\gamma_1 \\ \frac{1}{2}\triangle t(1-\gamma_1-\gamma_2) & 1-\triangle t(3+3\gamma_1+\gamma_2) & \frac{1}{2}\triangle t(1-\gamma_1-\gamma_2) \\ \frac{1}{2}\triangle t\gamma_1 & \frac{1}{2}\triangle t(1-\gamma_1-\gamma_2) & \frac{1}{2}\triangle t\gamma_1 \end{pmatrix},$$

we obtain the equations

$$\begin{aligned}
\gamma_1 &= \frac{a-2a^2}{1-3a+3a^2}, \\
\gamma_2 &= \frac{a^2}{1-3a+3a^2}, \quad \text{and} \\
\triangle t &= 2\left(a-3a^2+3a^3\right)
\end{aligned} \tag{3.29}$$

for $0 < a \leq \frac{1}{4}$.

Now we can derive the values of $\gamma_1$ and $\gamma_2$ by substituting $-1+3\gamma_1+9\gamma_2=0$ derived from (3.28) for the equations in (3.29)

$$\begin{aligned}
3\gamma_1+9\gamma_2 &= 1 \\
3\frac{a-2a^2}{1-3a+3a^2}+9\frac{a^2}{1-3a+3a^2} &= 1 \\
\frac{3a+3a^2}{1-3a+3a^2} &= 1 \\
a &= \frac{1}{6},
\end{aligned}$$

which leads to

$$\gamma_1 = \frac{4}{21}, \quad \gamma_2 = \frac{1}{21}, \quad \text{and} \quad \triangle t = \frac{7}{36}.$$

Consequently the determined values $\gamma_1 = \frac{4}{21}$ and $\gamma_2 = \frac{1}{21}$ induce the 3-D separable iteration kernel with the scale step $\triangle t = \frac{7}{36}$ and the Laplacian of the 3-D discrete scale-space kernel, respectively, given by

$$z \pm 1 : \begin{pmatrix} \frac{1}{216} & \frac{1}{54} & \frac{1}{216} \\ \frac{1}{54} & \frac{2}{27} & \frac{1}{54} \\ \frac{1}{216} & \frac{1}{54} & \frac{1}{216} \end{pmatrix}, \qquad z : \begin{pmatrix} \frac{1}{54} & \frac{2}{27} & \frac{1}{54} \\ \frac{2}{27} & \frac{8}{27} & \frac{2}{27} \\ \frac{1}{54} & \frac{2}{27} & \frac{1}{54} \end{pmatrix},$$

The 3-D iteration kernel

(3.30)

$$z \pm 1 : \begin{pmatrix} \frac{1}{21} & \frac{4}{21} & \frac{1}{21} \\ \frac{4}{21} & \frac{16}{21} & \frac{4}{21} \\ \frac{1}{21} & \frac{4}{21} & \frac{1}{21} \end{pmatrix}, \qquad z : \begin{pmatrix} \frac{4}{21} & \frac{16}{21} & \frac{4}{21} \\ \frac{16}{21} & -\frac{116}{21} & \frac{16}{21} \\ \frac{4}{21} & \frac{16}{21} & \frac{4}{21} \end{pmatrix}.$$

The Laplacian of the 3-D discrete scale-space kernel

# 4 Conclusion

In the first half part of this report we reviewed the discretization aspects of the scale-space theory as worked out by Lindeberg. The presented discrete theory is closely linked to the continuous scale-space theory through the discretization of the diffusion equation, where discrete nature is already taken into account in the formulation of the scale-space representation. While the 1-D discrete scale-space formulation derived by Lindeberg is theoretically obvious, his derivation for the higher dimensional discrete scale-space formulation has a few open questions as well as unclear points, which makes the extension to the higher dimensional discrete scale-space formulation relatively ambiguous.

Motivated by the open questions as well as unclear points in the higher dimensional (N-D) discrete scale-space formulation which were not explained by Lindeberg, we developed an improved discrete scale-space formulation for 2-D and 3-D signals based on a few assumptions, which was described in the last half part of this report. First, we defined the neighborhood connectivity and the Laplacian of the N-D discrete scale-space kernel. Then, on the basis of these definitions, we constructed the discretized diffusion equation, from which we derived the iteration kernel. Additionally, we imposed the restriction of the least possible rotational asymmetry on the discrete scale-space kernel, and we could determine the value of parameter $\gamma$ ($\gamma$ in 2-D and $\gamma_1$ and $\gamma_2$ in 3-D) which plays the role of preserving the rotational symmetry in the discrete scale-space kernel. Finally, we restricted the iteration kernel to be separable for the purpose of efficient computation. As a consequence, we derived the iteration kernel as well as the Laplacian of the discrete scale-space kernel of the form (3.24) in 2-D and those of the form (3.30) in 3-D.

By developing the improved discrete scale-space formulation for 2-D and 3-D signals, we made a remarkable step forward in investigating the matter of how to correctly approach the discrete scale-space theory. As a next step, based on the improved discrete scale-space formulation, we can identify routes to be taken for our possible future work as follows:

- The improved discrete scale-space kernel can be closely compared with the sampled Gaussian kernel which is commonly used for the discrete scale-space formulation (see e.g. Lim [10] for analysis of its problem).

- The improved discrete scale-space formulation can be applied to methods of scale selection, e.g., the BNS method, since the existing methods are based on the continuous scale-space theory.

- Lindeberg also suggested a scale selection method. Analytically or experimentally, we can compare the results of the discretized BNS method with that of the scale selection method by Lindeberg.

- Our improved discrete scale-space formulation may be compared to the discrete wavelet transform using derivatives of Gaussian wavelets.

- For specific test signals, for which a mathematically correct discretization can be given, a comparison of results for the continuous and discrete case, respectively, can be carried out.

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