The Supplemented Discrete Scale-Space Formulation

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March, 2002

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1 Introduction

The improved discrete scale-space (DSS) formulation for 2-D and 3-D signals developed in Lim [3] does not satisfy the semi-group property, so that it is necessary to supplement it.

In this report, first we look into the variance of the discrete scale-space kernels which is closely related to the semi-group property. Then, we address the problem of the improved discrete scale-space formulation, and suggest a generalized approach to supplementing it.

2 Variance of the DSS kernels

In this section, we describe how to calculate the variance of the DSS kernels based on the probability theory. The variance of the DSS kernel is important in order to examine whether the DSS kernel satisfies the semi-group property.

According to the probability theory ([1]), a probability function of discrete random variable X defined on a sample space is given by

$$f(x) = \begin{cases} P_i & \text{if } X = x_i \ (i = 1, 2, ...) \\ 0 \end{cases}.$$

For a given symmetric 1-D DSS kernel with the smallest variance derived in [3]

$$T(x) = \begin{pmatrix} \frac{1}{6} & \frac{2}{3} & \frac{1}{6} \end{pmatrix},$$

let us assume $T(x)$ to be a probability function $f(x)$ of the discrete random variable X with the possible values $x = -1, 0, 1$ given by

X	-1	0	1
$f(x)$	$\frac{1}{6}$	$\frac{2}{3}$	$\frac{1}{6}$

,

where the expected (or mean) value (as defined in [1]) is given by

$$E(X) = \sum_x x f(x) = -1 \cdot \frac{1}{6} + 0 \cdot \frac{2}{3} + 1 \cdot \frac{1}{6} = 0,$$

and the variance is given by

$$Var(X) = E(X^2) - [E(X)]^2 = (-1)^2 \cdot \frac{1}{6} + 0^2 \cdot \frac{2}{3} + 1^2 \cdot \frac{1}{6} - 0^2 = \frac{1}{3}.$$

On the other hand, the convolution of $T(x)$ with itself corresponds to

$$T(x) * T(x) = \begin{pmatrix} \frac{1}{36} & \frac{2}{9} & \frac{1}{2} & \frac{2}{9} & \frac{1}{36} \end{pmatrix},$$

where the mean is again 0 and the variance now is $\frac{2}{3}$ (i.e. $(-2)^2 \cdot \frac{1}{36} + (-1)^2 \cdot \frac{2}{9} + 0^2 \cdot \frac{1}{2} + 1^2 \cdot \frac{2}{9} + 2^2 \cdot \frac{1}{36} - 0^2$), from which one can see that the 1-D discrete scale-space kernel satisfies the semi-group property[1], since $h(\cdot;t_1) * h(\cdot;t_2) = h(\cdot;t_1+t_2)$. Accordingly, the 1-D DSS kernel derived from $k-1$ times convolution of $T(x)$ with itself is given by

$$T\left(x;\frac{k}{3}\right) = \underbrace{T\left(x;\frac{1}{3}\right) * \cdots * T\left(x;\frac{1}{3}\right)}_{k-1 \text{ times convolution}},$$

where $\frac{k}{3}$ is the variance.

In general, the N-D (or multivariate) normal density of the random vector is denoted by $N_N(\mu,\Sigma)$ ([2]), where μ $(N \times 1)$ is the expected (mean) vector and Σ $(N \times N)$ is the variance-covariance matrix e.g. in 3-D given by

$$\Sigma = \begin{pmatrix} \sigma_x^2 & \sigma_{xy}^2 & \sigma_{xz}^2 \\ \sigma_{yx}^2 & \sigma_y^2 & \sigma_{yz}^2 \\ \sigma_{zx}^2 & \sigma_{zy}^2 & \sigma_z^2 \end{pmatrix},$$

where σ_x^2, σ_y^2, and σ_z^2 correspond to the variances of the 1-D normal density of the random variables X, Y, and Z. If X, Y, and Z are independent random variables, then $E(XYZ) = E(X)E(Y)E(Z)$ holds. In general, the covariance of X and Y is given by $\sigma_{xy} = E(XY) - E(X)E(Y)$, and in case of independence $\sigma_{xy} = 0$ since $E(XY) = E(X)E(Y)$ (as is $\sigma_{xz} = 0$ and $\sigma_{yz} = 0$), which leads to

$$\Sigma = \begin{pmatrix} \sigma_x^2 & 0 & 0 \\ 0 & \sigma_y^2 & 0 \\ 0 & 0 & \sigma_z^2 \end{pmatrix}.$$

The N-D DSS kernel is given by

$$T(x_1, x_2, \cdot\cdot, x_N; t) = T(x_1;t) * T(x_2;t) * \cdot\cdot * T(x_N;t), \tag{2.1}$$

which is separable, and the variance t is identical with that of the convolved 1-D DSS kernels. Assuming that the N-D DSS kernel has been derived from the convolution of 1-D

[1]By definition of *group theory*, if every smoothing kernel is associated with a parameter value, and if two such kernels are convolved with each other, then the resulting kernel should be a member of the same family ([6]).

DSS kernel with itself as given in Eq. 2.1 yielding the multivariate normal density, the covariance matrix corresponds to

$$\Sigma_T = \begin{pmatrix} t & 0 & \cdots & 0 \\ 0 & t & \cdots & 0 \\ & & \cdot & \\ & & \cdot & \\ 0 & 0 & \cdots & t \end{pmatrix}.$$

Consequently, the variance of the N-D DSS kernel is calculated using the determinant of its covariance matrix such that

$$|\Sigma_T|^{\frac{1}{N}} = t.$$

3 Supplement for Higher Dimensional DSS Kernels

In Lim [3], we developed an improved discrete scale-space formulation for 2-D and 3-D signals based on the following (see [3] for details):

1. Definition of the neighborhood connectivity and of the Laplacian of the higher dimensional discrete scale-space kernel.

2. Construction of the discretized diffusion equation for the iteration kernel.

3. Determination of the parameter γ (in 3-D, γ_1 and γ_2) which plays the role of preserving the rotational symmetry of the discrete scale-space kernel based on the assumption that the iteration kernel should be separable.

However, the improved 2-D and 3-D discrete scale-space kernels derived in [3] do not satisfy the semi-group property which is one of the important properties of the linear scale-space theory. In this section, we clarify this problem, and suggest a generalized approach to supplementing it.

According to the semi-group property (see Footnote 1) and given a family of the scale-space representation L with increasing scale parameter t,

$$\begin{aligned} L(\cdot;t_1+t_2) &= L(\cdot;t_1) * G(\cdot;t_2) \\ &= f(\cdot) * G(\cdot;t_1) * G(\cdot;t_2) \end{aligned}$$

must hold, where $f(\cdot)$ corresponds to the original continuous signal and $G(\cdot;t)$ is the Gaussian kernel. Analogously, this property must be fulfilled in the discrete scale-space formulation, where the semi-group property means that the family of the scale-space representation is generated by

$$\begin{aligned} L(\cdot;k\cdot\triangle t) &= f(\cdot) * T(\cdot;k\cdot\triangle t) \\ &= f(\cdot) * (\underbrace{T(\cdot;\triangle t) * \cdots * T(\cdot;\triangle t)}_{k-1 \text{ times convolution}}), \end{aligned}$$

where $T(\cdot;\triangle t)$ is the discrete scale-space iteration kernel with scale step $\triangle t$, and the value of $\triangle t$ must be equivalent with that of the variance of T (such that t corresponds to the variance of $G(\cdot;t)$).

3.1 2-D Case

The value of γ is determined as $\frac{1}{5}$ and that of $\triangle t$ is as $\frac{5}{18}$ in [3], from which the 2-D iteration kernel is given by

$$T(x, y; \triangle t) = \begin{pmatrix} \frac{1}{36} & \frac{1}{9} & \frac{1}{36} \\ \frac{1}{9} & \frac{4}{9} & \frac{1}{9} \\ \frac{1}{36} & \frac{1}{9} & \frac{1}{36} \end{pmatrix}$$

(3.2)

$$= \begin{pmatrix} \frac{1}{6} & \frac{2}{3} & \frac{1}{6} \end{pmatrix} * \begin{pmatrix} \frac{1}{6} \\ \frac{2}{3} \\ \frac{1}{6} \end{pmatrix},$$

where $\triangle t = \frac{5}{18}$ and '*' denotes convolution. As mentioned in the previous section, the variance of T can be calculated using the determinant of its covariance matrix given by

$$|\Sigma_T|^{\frac{1}{2}} = \frac{1}{3},$$

where the covariance matrix of T is given by

$$\Sigma_T = \begin{pmatrix} \frac{1}{3} & 0 \\ 0 & \frac{1}{3} \end{pmatrix}.$$

One can see from above that the value of the variance (i.e. $|\Sigma_T|^{\frac{1}{2}} = \frac{1}{3}$) is not identical with the value of the scale step ($\triangle t = \frac{5}{18}$), which shows that the derived value of $\triangle t$ is incorrect. For example, according to the semi-group property, the 2-D DSS kernel derived from $k-1$ times self-convolution of $T(x, y; t)$ is given by

$$T(x, y; k \cdot t) = \underbrace{T(x, y; t) * \cdots * T(x, y; t)}_{k-1 \text{ times convolution}},$$

where

$$T(x, y; t) = \begin{pmatrix} \frac{1}{36} & \frac{1}{9} & \frac{1}{36} \\ \frac{1}{9} & \frac{4}{9} & \frac{1}{9} \\ \frac{1}{36} & \frac{1}{9} & \frac{1}{36} \end{pmatrix},$$

where t corresponds to $\frac{1}{3}$ (see above). Provided that $\triangle t = \frac{5}{18}$ is used as the variance value, the semi-group property cannot be satisfied; that is, $k \cdot \frac{5}{18}$ is far from being equivalent to $k \cdot \frac{1}{3}$ which is the correct variance value. And even worse, as k (i.e. the number of self-convolutions) increases, the difference between $k \cdot \frac{5}{18}$ and $k \cdot \frac{1}{3}$ increases.

As a consequence, we must correct the derivation made in [3]. Tracing back to the outset, we conjecture that there might exist a problem in the definition of the Laplacian of the 2-D discrete scale-space kernel (for detail see [3, Sec. 3.1.2]). In [3], for the definition of the Laplacian of ${}^2N_{\sqrt{2}}$, we intended to steer the ratio of rotational symmetry between the Laplacian of ${}^2N_1$ and the Laplacian of ${}^2N_{\sqrt{2}}$ only with the help of parameter γ, and thus we avoided setting an additional ambiguous (or unexplained) coefficient. In other words, we did not a priori define any coefficient, while Lindeberg set it to $\frac{1}{2}$. However, one can see clearly that this (i.e. avoidance of any coefficient in definition of the Laplacian of ${}^2N_{\sqrt{2}}$) does not give rise to the correct result (i.e. the 2-D DSS kernel does not satisfy the semi-group property). On the other hand, one cannot find any proof that the unexplained coefficient in the definition of the Laplacian of ${}^2N_{\sqrt{2}}$ given by Lindeberg is proper. Therefore, in a more generalized way, we set this coefficient as a variable λ and determine its proper value through the following theoretical derivation.

The 2-D discrete scale-space representation L satisfies

$$\partial_t L(x,y;t) = \frac{1}{2}\nabla^2 L(x,y;t) = \frac{1}{2}\left((1-\gamma)\nabla^2_{{}^2N_1} L(x,y;t) + \lambda\gamma\nabla^2_{{}^2N_{\sqrt{2}}} L(x,y;t)\right)$$

for $\gamma \in [0,1]$ and $\lambda \in (0,1)$, which can be further discretized using Euler's explicit method with scale step $\triangle t$ given by

(3.3)

$$\begin{aligned}
L^{k+1}_{x,y} =& L^k_{x,y} + \triangle t \left(\partial_t L^k_{x,y}\right) \\
=& L^k_{x,y} + \triangle t \frac{1}{2}\left((1-\gamma)\nabla^2_{{}^2N_1} L + \lambda\gamma\nabla^2_{{}^2N_{\sqrt{2}}} L\right) \\
=& (1 - 2\triangle t(1-\gamma+\lambda\gamma))L^k_{x,y} + \frac{1}{2}\triangle t(1-\gamma)\left(L^k_{x-1,y} + L^k_{x+1,y} + L^k_{x,y-1} + L^k_{x,y+1}\right) \\
& + \frac{\lambda}{2}\triangle t\gamma\left(L^k_{x-1,y-1} + L^k_{x-1,y+1} + L^k_{x+1,y-1} + L^k_{x+1,y+1}\right),
\end{aligned}$$

where subscripts x and y denote the spatial coordinates, and superscript k represents the iteration index. This discretization with scale step $\triangle t$ corresponds to the iteration with the 2-D discrete iteration kernel given by

$$T_{\triangle t} = \begin{pmatrix} \frac{\lambda}{2}\gamma\triangle t & \frac{1}{2}(1-\gamma)\triangle t & \frac{\lambda}{2}\gamma\triangle t \\ \frac{1}{2}(1-\gamma)\triangle t & 1-2\triangle t(1-\gamma+\lambda\gamma) & \frac{1}{2}(1-\gamma)\triangle t \\ \frac{\lambda}{2}\gamma\triangle t & \frac{1}{2}(1-\gamma)\triangle t & \frac{\lambda}{2}\gamma\triangle t \end{pmatrix}, \tag{3.4}$$

where the parameter γ plays the role of preserving the rotational symmetry of the 2-D discrete scale-space kernel.

The generating function describing one iteration of (3.3) is

$$^2\varphi_{step}(z,\chi) = (1-2\triangle t(1-\gamma+\lambda\gamma)) + \frac{(1-\gamma)}{2}\triangle tA + \frac{\lambda\gamma}{2}\triangle tB,$$

where

$$A = z^{-1} + z + \chi^{-1} + \chi$$
$$B = z^{-1}\chi^{-1} + z^{-1}\chi + z\chi^{-1} + z\chi,$$

and we obtain the generating function describing the composed transformation ($\triangle t = \frac{t}{n}$) as

$$\begin{aligned} ^2\varphi_{composed,n}(z,\chi) &= \left(^2\varphi_{step}(z,\chi)\right)^n \\ &= \left(1 + \frac{t}{n}\left(-2(1-\gamma+\lambda\gamma) + \frac{(1-\gamma)}{2}A + \frac{\lambda\gamma}{2}B\right)\right)^n. \end{aligned}$$

Based on the fact that $\lim_{n\to\infty}(1+\alpha_n/n)^n = e^\alpha$ if $\lim_{n\to\infty}\alpha_n = \alpha$, the generating function of the kernel describing the transformation from the original signal to the representation at a certain scale t is given by

$$\begin{aligned} ^2\varphi_T(z,\chi) &= \sum_{(m,n)\in\mathbb{Z}^2} T(m,n;t)z^m\chi^n \\ &= e^{t\left(-2(1-\gamma+\lambda\gamma)+\frac{(1-\gamma)}{2}A+\frac{\lambda\gamma}{2}B\right)}. \end{aligned}$$

Its Fourier transform is derived by replacing complex variables z and χ with e^{-iu} and e^{-iv}

as

$$\begin{aligned}\mathcal{F}\left({}^2\varphi_T(z,\chi)\right) &= {}^2\psi_T(e^{-iu}, e^{-iv}) \\ &= {}^2\psi_T(\cos u - i\sin u, \cos v - i\sin v) \\ &= e^{t(-2(1-\gamma+\lambda\gamma)+(1-\gamma)(\cos u+\cos v)+\lambda\gamma 2\cos u\cos v)},\end{aligned}$$

which can be transformed into polar coordinates given a fixed value of radius r and an angular variable ϕ such that $u = r\cos\phi$ and $v = r\sin\phi$:

$${}^2\psi_T(r,\phi) = e^{(t\cdot k(r,\phi))},$$

where

$$\begin{aligned}k(r,\phi) = -2(1-\gamma+\lambda\gamma) + (1-\gamma)(\cos(r\cos\phi) + \cos(r\sin\phi)) \\ + \lambda\gamma 2\cos(r\cos\phi)\cos(r\sin\phi).\end{aligned}$$

$k(r,\phi)$ depends on three variables, i.e., r, ϕ, and γ, and we intend to determine the value of γ which gives the smallest angular variation of ϕ for a fixed value r. For examining the ϕ-dependency of γ from $k(r,\phi)$, we expand the MacLaurin series of $k(r,\phi)$ with respect to r with the help of Mathematica ([7]);

$$\begin{aligned}k(r,\phi) &= -\frac{1-\gamma+2\lambda\gamma}{2}r^2 + O(r^4) \\ &= -\frac{1-\gamma+2\lambda\gamma}{2}r^2 + \frac{3+3\gamma(4\lambda-1)+(1-\gamma-4\lambda\gamma)\cos 4\phi}{96}r^4 + O(r^6).\end{aligned}$$

It is clear from the MacLaurin series that if $1-\gamma-4\lambda\gamma = 0$ then the ϕ-dependency decreases by r^6 instead of r^4, which implies that the smallest angular variation is given with respect to the 2-D discrete scale-space kernel when $\gamma = 1/(1+4\lambda)$. That is to say, $\gamma = 1/(1+4\lambda)$ yields the least possible rotational asymmetry for the 2-D discrete scale-space kernel.

As a consequence, the 2-D iteration kernel of (3.4) by substituting $\gamma = 1/(1+4\lambda)$ corresponds to

$$T_{\triangle t} = \begin{pmatrix} \frac{\lambda}{2(1+4\lambda)}\triangle t & \frac{2\lambda}{1+4\lambda}\triangle t & \frac{\lambda}{2(1+4\lambda)}\triangle t \\ \frac{2\lambda}{1+4\lambda}\triangle t & 1-\frac{10\lambda}{1+4\lambda}\triangle t & \frac{\lambda}{2(1+4\lambda)}\triangle t \\ \frac{\lambda}{2(1+4\lambda)}\triangle t & \frac{2\lambda}{1+4\lambda}\triangle t & \frac{\lambda}{2(1+4\lambda)}\triangle t \end{pmatrix}, \tag{3.5}$$

where $\triangle t > 0$.

According to the assumption that the 2-D iteration kernel should be separable (i.e., the 2-D iteration kernel should be constructed by convolution of the 1-D kernel with itself),

$$\begin{pmatrix} a & 1-2a & a \end{pmatrix} * \begin{pmatrix} a \\ 1-2a \\ a \end{pmatrix} = \begin{pmatrix} a^2 & a(1-2a) & a^2 \\ a(1-2a) & (1-2a)^2 & a(1-2a) \\ a^2 & a(1-2a) & a^2 \end{pmatrix}$$
$$= \begin{pmatrix} \frac{\lambda}{2(1+4\lambda)}\triangle t & \frac{2\lambda}{1+4\lambda}\triangle t & \frac{\lambda}{2(1+4\lambda)}\triangle t \\ \frac{2\lambda}{1+4\lambda}\triangle t & 1-\frac{10\lambda}{1+4\lambda}\triangle t & \frac{\lambda}{2(1+4\lambda)}\triangle t \\ \frac{\lambda}{2(1+4\lambda)}\triangle t & \frac{2\lambda}{1+4\lambda}\triangle t & \frac{\lambda}{2(1+4\lambda)}\triangle t \end{pmatrix},$$

for $0 < a \leq \frac{1}{4}$, we obtain

$$a = \frac{1}{6} \quad \text{and} \quad \triangle t = \frac{2}{9} + \frac{1}{18\lambda}.$$

Besides, in order to satisfy the semi-group property, $\triangle t$ should correspond to the variance of the 2-D discrete iteration kernel (see Section 2), which means

$$\triangle t = 2a$$

must hold. Then, we finally have

$$a = \frac{1}{6}, \quad \lambda = \frac{1}{2}, \quad \gamma = \frac{1}{3}, \quad \text{and} \quad \triangle t = \frac{1}{3}.$$

It is noticeable here that the value of parameter λ (i.e. $\frac{1}{2}$) that corresponds to the coefficient in the definition of the Laplacian of ${}^2N_{\sqrt{2}}$ is equal to that defined by Lindeberg in [5, p. 105].

Consequently, the 2-D separable iteration kernel for the rotationally least asymmetric 2-D discrete scale-space kernel satisfying the semi-group property is given by

$$T_{\triangle t} = \begin{pmatrix} \frac{1}{6} & \frac{2}{3} & \frac{1}{6} \end{pmatrix} * \begin{pmatrix} \frac{1}{6} \\ \frac{2}{3} \\ \frac{1}{6} \end{pmatrix} = \begin{pmatrix} \frac{1}{36} & \frac{1}{9} & \frac{1}{36} \\ \frac{1}{9} & \frac{4}{9} & \frac{1}{9} \\ \frac{1}{36} & \frac{1}{9} & \frac{1}{36} \end{pmatrix}, \tag{3.6}$$

where $\triangle t = \frac{1}{3}$.

3.2 3-D Case

In [3], for the least rotationally asymmetric 3-D discrete scale-space kernel, parameters γ_1 and γ_2 are determined as $\frac{4}{21}$ and $\frac{1}{21}$, and the 3-D discrete iteration kernel is given by

$$
\begin{aligned}
T(x,y,z;\triangle t) &= \begin{pmatrix} \frac{1}{216} & \frac{1}{54} & \frac{1}{216} \\ \frac{1}{54} & \frac{2}{27} & \frac{1}{54} \\ \frac{1}{216} & \frac{1}{54} & \frac{1}{216} \end{pmatrix}_{z\pm1} , \begin{pmatrix} \frac{1}{54} & \frac{2}{27} & \frac{1}{54} \\ \frac{2}{27} & \frac{8}{27} & \frac{2}{27} \\ \frac{1}{54} & \frac{2}{27} & \frac{1}{54} \end{pmatrix}_{z} \\
&= \begin{pmatrix} \frac{1}{6} & \frac{2}{3} & \frac{1}{6} \end{pmatrix}_x * \begin{pmatrix} \frac{1}{6} & \frac{2}{3} & \frac{1}{6} \end{pmatrix}_y * \begin{pmatrix} \frac{1}{6} & \frac{2}{3} & \frac{1}{6} \end{pmatrix}_z
\end{aligned}
\tag{3.7}
$$

where $\triangle t = \frac{7}{36}$.

The variance of T of Eq. 3.7 is calculated using the determinant of its covariance matrix (equivalently to the 2-D case; see Section 2) given by

$$|\Sigma_T|^{\frac{1}{3}} = \frac{1}{3},$$

where

$$\Sigma_T = \begin{pmatrix} \frac{1}{3} & 0 & 0 \\ 0 & \frac{1}{3} & 0 \\ 0 & 0 & \frac{1}{3} \end{pmatrix}.$$

Similarly to the 2-D case, the variance ($T = \frac{1}{3}$) is not equivalent to the scale step ($\triangle t = \frac{7}{36}$), which implies that the semi-group property cannot be satisfied (see Section 3.1).

As already shown for the the 2-D case in the previous section, we here correct the definition of the Laplacians of ${}^3N_{\sqrt{2}}$ and of ${}^3N_{\sqrt{3}}$ set as in [3]. In defining the Laplacians of ${}^3N_{\sqrt{2}}$ and of ${}^3N_{\sqrt{3}}$, we intended to steer the ratio of rotational symmetry between the Laplacian of ${}^3N_1$, the Laplacian of ${}^3N_{\sqrt{2}}$, and the Laplacian of ${}^3N_{\sqrt{3}}$ only with the help of parameters γ_1 and γ_2. Therefore, we avoided setting additional ambiguous (or unexplained) coefficients or, respectively, we did not a priori define any coefficients in the definition of the Laplacians of ${}^3N_{\sqrt{2}}$ and of ${}^3N_{\sqrt{3}}$, while Lindeberg set them both to $\frac{1}{4}$. However, our outset does not give rise to the correct result (i.e., the semi-group property is not satisfied). On the other hand, it is not clear why those parameters set by Lindeberg both must be $\frac{1}{4}$.

As a consequence, we again approach this problem in a more generalized way: We define the coefficients of the Laplacians of ${}^3N_{\sqrt{2}}$ and of ${}^3N_{\sqrt{3}}$ as λ_1 and λ_2, and determine their proper values through the following theoretical derivation.

The 3-D discrete scale-space representation L satisfies

$$\partial_t L = \frac{1}{2}\left((1-\gamma_1-\gamma_2)\nabla^2_{{}^3N_1}L + \lambda_1\gamma_1\nabla^2_{{}^3N_{\sqrt{2}}}L + \lambda_2\gamma_2\nabla^2_{{}^3N_{\sqrt{3}}}L\right)$$

for $\gamma_1, \gamma_2 \in [0,1]$ and $\lambda_1, \lambda_2 \in (0,1)$, which can be discretized using Euler's explicit method with scale step $\triangle t$ given by

(3.8)

$$\begin{aligned}
L^{k+1}_{x,y,z} =& L^k_{x,y,z} + \triangle t\left(\partial_t L^k_{x,y,z}\right)\\
=& L^k_{x,y,z} + \triangle t\frac{1}{2}\left((1-\gamma_1-\gamma_2)\nabla^2_{{}^3N_1}L + \lambda_1\gamma_1\nabla^2_{{}^3N_{\sqrt{2}}}L + \lambda_2\gamma_2\nabla^2_{{}^3N_{\sqrt{3}}}L\right)\\
=& (1+\triangle t(-3+(3-6\lambda_1)\gamma_1+(3-4\lambda_2)\gamma_2))L^k_{x,y,z} +\\
& \frac{1}{2}\triangle t(1-\gamma_1-\gamma_2)(L^k_{x,y,z-1} + L^k_{x,y,z+1} + L^k_{x,y-1,z} + L^k_{x,y+1,z} + L^k_{x-1,y,z} + L^k_{x+1,y,z}) +\\
& \frac{\lambda_1}{2}\triangle t\gamma_1(L^k_{x-1,y-1,z} + L^k_{x-1,y+1,z} + L^k_{x+1,y-1,z} + L^k_{x+1,y+1,z} + L^k_{x-1,y,z-1} + L^k_{x-1,y,z+1} +\\
& L^k_{x+1,y,z-1} + L^k_{x-1,y,z-1} + L^k_{x,y-1,z-1} + L^k_{x,y-1,z+1} + L^k_{x,y+1,z-1} + L^k_{x,y+1,z+1}) +\\
& \frac{\lambda_2}{2}\triangle t\gamma_2(L^k_{x-1,y-1,z-1} + L^k_{x+1,y-1,z-1} + L^k_{x-1,y+1,z-1} + L^k_{x-1,y-1,z+1} + L^k_{x-1,y+1,z+1} +\\
& L^k_{x+1,y-1,z+1} + L^k_{x+1,y+1,z-1} + L^k_{x+1,y+1,z+1}),
\end{aligned}$$

where subscripts x, y, and z denote the spatial coordinates, and superscript k represents the iteration index. This discretization with $\triangle t$ corresponds to the iteration with the 3-D discrete iteration kernel described with respect to $z \pm 1$ and z along the z-axis given by

(3.9)

$$T_{\triangle t} = \begin{pmatrix} \frac{\lambda_2}{2}\triangle t\gamma_2 & \frac{\lambda_1}{2}\triangle t\gamma_1 & \frac{\lambda_2}{2}\triangle t\gamma_2 \\ \frac{\lambda_1}{2}\triangle t\gamma_1 & \frac{1}{2}\triangle t(1-\gamma_1-\gamma_2) & \frac{\lambda_1}{2}\triangle t\gamma_1 \\ \frac{\lambda_2}{2}\triangle t\gamma_2 & \frac{\lambda_1}{2}\triangle t\gamma_1 & \frac{\lambda_2}{2}\triangle t\gamma_2 \end{pmatrix}_{z\pm1}$$

$$\begin{pmatrix} \frac{\lambda_1}{2}\triangle t\gamma_1 & \frac{1}{2}\triangle t(1-\gamma_1-\gamma_2) & \frac{\lambda_1}{2}\triangle t\gamma_1 \\ \frac{1}{2}\triangle t(1-\gamma_1-\gamma_2) & 1+\triangle t(-3+(3-6\lambda_1)\gamma_1+(3-4\lambda_2)\gamma_2) & \frac{1}{2}\triangle t(1-\gamma_1-\gamma_2) \\ \frac{\lambda_1}{2}\triangle t\gamma_1 & \frac{1}{2}\triangle t(1-\gamma_1-\gamma_2) & \frac{\lambda_1}{2}\triangle t\gamma_1 \end{pmatrix}_z,$$

where the parameters γ_1 and γ_2 play the role of preserving the rotational symmetry of the 3-D discrete scale-space kernel.

The generating function describing one iteration of (3.8) is

$$\begin{aligned}{}^3\varphi_{step}(z,\chi,\tau) = &(1+\triangle t(-3+(3-6\lambda_1)\gamma_1+(3-4\lambda_2)\gamma_2))+\\ &\frac{1}{2}\triangle t(1-\gamma_1-\gamma_2)A+\frac{\lambda_1}{2}\triangle t\gamma_1 B+\frac{\lambda_2}{2}\triangle t\gamma_2 C,\end{aligned}$$

where

$$\begin{aligned}A &: z^{-1}+z+\chi^{-1}+\chi+\tau^{-1}+\tau\\ B &: z^{-1}\chi^{-1}+z^{-1}\chi+z\chi^{-1}+z\chi+z^{-1}\tau^{-1}+z^{-1}\tau+z\tau^{-1}+z\tau+\chi^{-1}\tau^{-1}+\chi^{-1}\tau+\chi\tau^{-1}+\chi\tau\\ C &: z^{-1}\chi^{-1}\tau^{-1}+z\chi^{-1}\tau^{-1}+z^{-1}\chi\tau^{-1}+z^{-1}\chi^{-1}\tau+z^{-1}\chi\tau+z\chi^{-1}\tau+z\chi\tau^{-1}+z\chi\tau,\end{aligned}$$

and we obtain the generating function describing the composed transformation ($\triangle t = \frac{t}{n}$)

$$\begin{aligned}{}^3\varphi_{composed,n}(z,\chi,\tau) = \left({}^3\varphi_{step}(z,\chi,\tau)\right)^n =\\ \left(1+\frac{t}{n}\left((-3+(3-6\lambda_1)\gamma_1+(3-4\lambda_2)\gamma_2)+\frac{(1-\gamma_1-\gamma_2)}{2}A+\frac{\lambda_1\gamma_1}{2}B+\frac{\lambda_2\gamma_2}{2}C\right)\right)^n.\end{aligned}$$

Based on the fact that $\lim_{n\to\infty}(1+\alpha_n/n)^n = e^{\alpha}$ if $\lim_{n\to\infty}\alpha_n = \alpha$, the generating function of the kernel describing the transformation from the original signal to the representation at a certain scale t is given by

$$\begin{aligned}{}^3\varphi_T(z,\chi,\tau) = &\sum_{(m,n,l)\in\mathbb{Z}^3} T(m,n,l;t)z^m\chi^n\tau^l\\ =&e^{t((-3+(3-6\lambda_1)\gamma_1+(3-4\lambda_2)\gamma_2)+\frac{(1-\gamma_1-\gamma_2)}{2}A+\frac{\lambda_1\gamma_1}{2}B+\frac{\lambda_2\gamma_2}{2}C)}.\end{aligned}$$

Its Fourier transform is derived by replacing complex variables z, χ, and τ, respectively, with e^{-iu}, e^{-iv}, and $e^{-i\omega}$ as

$$\begin{aligned}&\mathcal{F}\left({}^3\varphi_T(z,\chi,\tau)\right) = {}^3\psi_T(e^{-iu},e^{-iv},e^{-i\omega}) = {}^3\psi_T(\cos u - i\sin u, \cos v - i\sin v, \cos\omega - i\sin\omega)\\ &= e^{t((-3+(3-6\lambda_1)\gamma_1+(3-4\lambda_2)\gamma_2)+(1-\gamma_1-\gamma_2)(\cos u+\cos v+\cos\omega)+\lambda_1\gamma_1 2(\cos u\cos v+\cos u\cos\omega+\cos v\cos\omega)+\lambda_2\gamma_2 4\cos u\cos v\cos\omega)},\end{aligned}$$

which can be transformed into spherical coordinates as a function of spherical angles ϕ and θ given a fixed value of radius r such that $u = r\cos\phi\sin\theta$, $v = r\sin\phi\sin\theta$, and $\omega = r\cos\theta$:

$${}^3\psi_T(r,\phi,\theta) = e^{(t\cdot k(r,\phi,\theta))},$$

where

$$\begin{aligned}
k(r,\phi,\theta) &= (-3+(3-6\lambda_1)\gamma_1+(3-4\lambda_2)\gamma_2)+\\
&\quad (1-\gamma_1-\gamma_2)(\cos(r\cos\phi\sin\theta)+\cos(r\sin\phi\sin\theta)+\cos(r\cos\theta))+\\
&\quad \lambda_1\gamma_1 2(\cos(r\cos\phi\sin\theta)\cos(r\sin\phi\sin\theta)+\cos(r\cos\phi\sin\theta)\cos(r\cos\theta)+\\
&\qquad\quad \cos(r\sin\phi\sin\theta)\cos(r\cos\theta))+\\
&\quad \lambda_2\gamma_2 4\cos(r\cos\phi\sin\theta)\cos(r\sin\phi\sin\theta)\cos(r\cos\theta).
\end{aligned}$$

Now from $k(r,\phi,\theta)$ we must determine the values of γ_1 and γ_2 that give the smallest angular variation of ϕ and θ for a fixed value r. For examining the ϕ- and θ-dependency of γ_1 and γ_2 from $k(r,\phi,\theta)$, we expand the MacLaurin series of $k(r,\phi,\theta)$ with respect to r with the help of Mathematica ([7]);

$$\begin{aligned}
& k(r,\phi,\theta)\\
= \quad & O(r^4)-\frac{1-\gamma_1+4\lambda_1\gamma_1-\gamma_2+4\lambda_2\gamma_2}{2}r^2\\
= \quad & O(r^6)-\frac{1-\gamma_1+4\lambda_1\gamma_1-\gamma_2+4\lambda_2\gamma_2}{2}r^2+\\
& \Big((1+(-1+4\lambda_1)\gamma_1-\gamma_2+4\lambda_2\gamma_2)\cos^4(\theta)+12(\lambda_1\gamma_1+2\lambda_2\gamma_2)\cos^2(\theta)\sin^2(\theta)-\\
& (3(-1+\gamma_1-6\lambda_1\gamma_1+\gamma_2-8\lambda_2\gamma_2)+(-1+\gamma_1+2\lambda_1\gamma_1+\gamma_2+8\lambda_2\gamma_2)\cos(4\phi))\frac{\sin^4(\theta)}{4}\Big)\frac{r^4}{24},
\end{aligned}$$

where the ϕ-dependency of $k(r,\phi,\theta)$ decreases by r^6 instead of r^4 if

$$-1+\gamma_1+2\lambda_1\gamma_1+\gamma_2+8\lambda_2\gamma_2=0,$$

and the θ-dependency of $k(r,\phi,\theta)$ decreases by r^6 instead of r^4 if

$$\begin{aligned}
1+(-1+4\lambda_1)\gamma_1-\gamma_2+4\lambda_2\gamma_2 &= 1,\\
12(\lambda_1\gamma_1+2\lambda_2\gamma_2) &= 2, \qquad \text{and}\\
-\frac{3(-1+\gamma_1-6\lambda_1\gamma_1+\gamma_2-8\lambda_2\gamma_2)}{4} &= 1,
\end{aligned}$$

from which follows

$$k(r,\phi,\theta)=-\frac{r^2}{2}+\frac{r^4}{24}+O(r^6).$$

This implies that the smallest angular variation is given for the 3-D discrete scale-space kernel if the following terms are satisfied

$$
\begin{aligned}
-1+\gamma_1+2\lambda_1\gamma_1+\gamma_2+8\lambda_2\gamma_2 &= 0,\\
1+(-1+4\lambda_1)\gamma_1-\gamma_2+4\lambda_2\gamma_2 &= 1,\\
12(\lambda_1\gamma_1+2\lambda_2\gamma_2) &= 2, \qquad \text{and}\\
-\frac{3(-1+\gamma_1-6\lambda_1\gamma_1+\gamma_2-8\lambda_2\gamma_2)}{4} &= 1,
\end{aligned}
$$

from which we obtain with the help of Mathematica

$$
\lambda_1=\frac{-1+3\gamma_1+3\gamma_2}{6\gamma_1}, \qquad \lambda_2=\frac{2-3\gamma_1-3\gamma_2}{12\gamma_2}. \tag{3.10}
$$

On the other hand, according to the assumption of separability of the iteration kernel, $T_{\triangle t}$ of (3.9) should be separable

$$
T_{\triangle t}=\begin{pmatrix}a & 1-2a & a\end{pmatrix}_x * \begin{pmatrix}a & 1-2a & a\end{pmatrix}_y * \begin{pmatrix}a & 1-2a & a\end{pmatrix}_z
$$

for $0<a\leq\frac{1}{4}$, from which follows

$$
z\pm 1: \begin{pmatrix} a^3 & a^2(1-2a) & a^3\\ a^2(1-2a) & a(1-2a)^2 & a^2(1-2a)\\ a^3 & a^2(1-2a) & a^3 \end{pmatrix} = \begin{pmatrix} \frac{\lambda_2}{2}\triangle t\gamma_2 & \frac{\lambda_1}{2}\triangle t\gamma_1 & \frac{\lambda_2}{2}\triangle t\gamma_2\\ \frac{\lambda_1}{2}\triangle t\gamma_1 & \frac{1}{2}\triangle t(1-\gamma_1-\gamma_2) & \frac{\lambda_1}{2}\triangle t\gamma_1\\ \frac{\lambda_2}{2}\triangle t\gamma_2 & \frac{\lambda_1}{2}\triangle t\gamma_1 & \frac{\lambda_2}{2}\triangle t\gamma_2 \end{pmatrix},
$$

$$
z: \begin{pmatrix} a^2(1-2a) & a(1-2a)^2 & a^2(1-2a)\\ a(1-2a)^2 & (1-2a)^3 & a(1-2a)^2\\ a^2(1-2a) & a(1-2a)^2 & a^2(1-2a) \end{pmatrix} =
$$

$$
\begin{pmatrix} \frac{\lambda_1}{2}\triangle t\gamma_1 & \frac{1}{2}\triangle t(1-\gamma_1-\gamma_2) & \frac{\lambda_1}{2}\triangle t\gamma_1\\ \frac{1}{2}\triangle t(1-\gamma_1-\gamma_2) & 1+\triangle t(-3+(3-6\lambda_1)\gamma_1+(3-4\lambda_2)\gamma_2) & \frac{1}{2}\triangle t(1-\gamma_1-\gamma_2)\\ \frac{\lambda_1}{2}\triangle t\gamma_1 & \frac{1}{2}\triangle t(1-\gamma_1-\gamma_2) & \frac{\lambda_1}{2}\triangle t\gamma_1 \end{pmatrix},
$$

and $\triangle t$ must be equal to $2a$ (i.e., $\triangle t$ has to correspond to the variance of the 3-D discrete iteration kernel; see Section 2) in order to satisfy the semi-group property, from which we have

$$\lambda_1 = \frac{a - 2a^2}{r1},$$
$$\lambda_2 = \frac{a^2}{4a - 4a^2 - \gamma_1}, \tag{3.11}$$
$$\gamma_2 = 4a - 4a^2 - \gamma_1$$

for $0 < a \leq \frac{1}{4}$. Now we can find the parameter values by solving the equations in (3.10) and in (3.11) as

$$\begin{aligned} a &= \frac{1}{6}, & \triangle t &= \frac{1}{3}, \\ \gamma_1 &= \frac{4}{9}, & \gamma_2 &= \frac{1}{9}, \\ \lambda_1 &= \frac{1}{4}, & \lambda_2 &= \frac{1}{4}. \end{aligned}$$

The values of parameters λ_1 and λ_2 (i.e., $\frac{1}{4}$ and $\frac{1}{4}$) which correspond to the coefficients of the Laplacian of ${}^3N_{\sqrt{2}}$ and of ${}^3N_{\sqrt{3}}$ are identical with those defined by Lindeberg in [5, p. 105].

Consequently, the 3-D separable iteration kernel for the rotationally least asymmetric 3-D discrete scale-space kernel satisfying the semi-group property is given by

$$\begin{aligned} T_{\triangle}t &= \begin{pmatrix} \frac{1}{6} & \frac{2}{3} & \frac{1}{6} \end{pmatrix}_x * \begin{pmatrix} \frac{1}{6} & \frac{2}{3} & \frac{1}{6} \end{pmatrix}_y * \begin{pmatrix} \frac{1}{6} & \frac{2}{3} & \frac{1}{6} \end{pmatrix}_z \\ &= \begin{pmatrix} \frac{1}{216} & \frac{1}{54} & \frac{1}{216} \\ \frac{1}{54} & \frac{2}{27} & \frac{1}{54} \\ \frac{1}{216} & \frac{1}{54} & \frac{1}{216} \end{pmatrix}_{z\pm 1}, \quad \begin{pmatrix} \frac{1}{54} & \frac{2}{27} & \frac{1}{54} \\ \frac{2}{27} & \frac{8}{27} & \frac{2}{27} \\ \frac{1}{54} & \frac{2}{27} & \frac{1}{54} \end{pmatrix}_z, \end{aligned} \tag{3.12}$$

where $\triangle t = \frac{1}{3}$.

4 Conclusion

Motivated by the open questions and unclear points in the higher dimensional discrete scale-space formulation by Lindeberg ([4], [5]), we developed the improved discrete scale-space formulation for 2-D and 3-D signals based on a few assumptions as given in [3].

However, the derived 2-D and 3-D discrete scale-space kernels in [3] do not satisfy the semi-group property since the definition of the higher dimensional Laplacian is not proper. In this work, we derived the supplemented discrete scale-space formulation which satisfies the semi-group property. Through a generalized theoretical derivation, it becomes clear that the coefficient in the definition of the Laplacian of ${}^2N_{\sqrt{2}}$ in 2-D must be set to $\frac{1}{2}$, whereas those in the definition of the Laplacians of ${}^3N_{\sqrt{2}}$ and ${}^3N_{\sqrt{3}}$ must be set both to $\frac{1}{4}$. That is to say, the 2-D and 3-D discrete scale-space kernels satisfying the semi-group property can be obtained from the discrete scale-space formulation derived in [3], if the coefficients in the definition of the Laplacians are corrected.

Although the coefficient values (i.e., $\frac{1}{2}$ in 2-D as well as $\frac{1}{4}$ and $\frac{1}{4}$ in 3-D) from our theoretical derivation are identical with those defined by Lindeberg [5, p. 105], the result in this work proves why those coefficients must be set as such in the definition of the Laplacians-a result which one cannot find in Lindeberg's work (e.g. [4] or [5]).

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