Grid-Independent Necessary Criterions for Shape Preserving Digitization

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ABSTRACT

The question, which shapes can be digitized without any change in the fundamental geometric and topological properties is of great importance for the reliability of image analysis algorithms, but is nevertheless unanswered for a lot of digitization schemes. While r-regularity is a sufficient criterion for shapes to be reconstructed correctly by using any regular or irregular sampling grid of certain density, necessary criteria are up to now unknown.

The author proves such a necessary criterion: If you choose some sampling grid and you want a shape to be digitized correctly with any alignment of this grid, then the shape has to be a bordered 2D-manifold, i.e. its boundary has to have no junctions. This implies that any correct digitization is an extended well-composed set and thus the well known problems of defining connectivity in 2D are always due to wrong sampling or improper original shapes. This is of great importance, since extended well-composed sets have many nice topological properties, for example the Jordan curve theorem holds and the Euler characteristic is locally computable.

Moreover the author proves a second necessary criterion: In case of a correct digitization with a grid of a certain density, shapes are not allowed to have corners with an angle smaller than 60 degrees. In case of common square grids the smallest possible angle is even 90 degrees. If some shape has some corner with a too small angle, the shape can not be digitized topologically correctly with every alignment of some sampling grid, if this grid exceeds a certain density. Thus the intuitive assumption that a finer grid would lead to a better digitization (in a topological sense) is simply wrong.

Keywords: sampling, digitization, shape preservation, manifolds, Jordan curve theorem, Euler characteristic

1. INTRODUCTION

The process of digitization is one of the basics of digital image processing. The aim of image analysis is to derive statements about the real world by looking at some digitized part of it. Therefore it is important to understand, which information gets lost during digitization. But even in the case of sampling binary images a lot of questions are still unanswered. It is known, that a certain subclass of binary images, known as r-regular images, can be sampled and reconstructed without changes in topology.[1–3] It is also known that a shape, which can be digitized correctly with any sampling grid of a certain density, has to be r-regular.[4] But normally one uses a specific grid type (e.g. a square grid) and thus wants to know, which shapes can be sampled correctly with this grid. This paper gives a necessary criterion for these shapes, which is powerful enough to have direct consequences for the properties of digital images in general. It is shown that both continuous and digital shapes have to be bounded 2D-manifolds (in case of square grids this equals wellcomposed sets in the sense of Latecki[5]), which solves several digital connectivity problems and makes it possible to derive a discrete Jordan curve theorem and to compute the Euler characteristic locally for any sampling grid type.

Moreover the author proves that shapes are not allowed to have corners with angles smaller than $\frac{\pi}{3}$ (i.e. 60 degrees), and in case of square grids even any acute angle (i.e. smaller than 90 degrees) is not allowed if one wants to guarantee the digitization to be topologically correctly.

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2. DEFINITION OF DIGITIZATION AND PRIOR RESULTS

In order to make statements about necessary criterions for shape preserving digitization, the process of digitization has to be formalized. At first a definition of the sampling grid and the pixels is needed. In general a sampling grid is a countable subset of $\mathbb{R}^2$, such that the points are not too sparse or too dense anywhere. This definition is often used when no restriction to a certain grid type is wanted.[3,4,6] Then the simplest idea to define the pixels is to choose the Voronoi regions of the sampling grid.

DEFINITION 2.1. *A countable set $S \subset \mathbb{R}^2$ of* sampling points*, where the Euclidean distance from each point $x \in \mathbb{R}^2$ to the next sampling point is at most $r' \in \mathbb{R}$, is called an r'*-grid *if $S \cap A$ is finite for any bounded set $A \in \mathbb{R}^2$. An r'-grid is also called* sampling grid*. The* pixel $\text{Pixel}_S(s)$ *of a sampling point s is its Voronoi region, i.e. the set of all points lying at least as near to this point than to any other sampling point. A point $c \in \mathbb{R}^2$ being element of at least three pixels, is called* corner point*. The union of the pixels with sampling points lying in A is the* digital reconstruction of A w.r.t. S*: $\hat{A} := \bigcup_{s \in S \cap A} \text{Pixel}_S(s)$.*

Due to the definition the digital reconstruction is simply the union of pixels whose sampling points lie in the specified set. This is equal to the digitization model used by Pavlidis[1] and Serra.[2] Other digitization schemes like the subset digitization and the v-digitization[5,7] can be interpreted as the chosen digitization of some blurred image with additional thresholding.[3,4]

In order to justify if a digital reconstruction is a good reconstruction of the given set, a method to compare these sets is necessary. Since we are interested in topological properties, the two sets should be topologically the same. In two dimensions this means they have to be $\mathbb{R}^2$-homeomorphic.[4] This means, there exists a bijective function $f : \mathbb{R}^2 \to \mathbb{R}^2$ (called $\mathbb{R}^2$-homeomorphism) mapping the original set onto its digital reconstruction, such that both f and f^{-1} are continuous.

There are several sufficient criterions known for shape preserving digitization. Most of them are based on so-called r-regular sets. E.g. Pavlidis showed that r-regular sets can be digitized with any square grid of a certain density whithout topology changes.[1] Serra showed the same for hexagonal grids[2] and recently U. Köthe and the author generalized these results to arbitrary grid types.[3,4] r-regular sets have a restriction to the curvature of their boundary. Thus neither corners nor junctions can occur. This is very restrictive since corners and junctions play an important role in image analysis. Thus the aim of this paper is to investigate if the absense of corners and junctions is not only sufficient but also necessary for a shape preserving digitization.

3. DIGITIZATION OF NON-MANIFOLD SHAPES

A bordered 2D manifold is a set where for each boundary point there exists a neighborhood which is homeomorphic to a closed half-plane. So, if a set is a bordered 2D-manifold it has no junctions. Thus at first a formal method to identify junctions on the boundary of the set is needed. Informally a junction occurs where more than two boundary lines meet. This number of boundary lines is called the local junction number:

DEFINITION 3.1. *Let $A \subset \mathbb{R}^2$ be a set and $x \in \mathbb{R}^2$ be a point and let $\mathcal{B}_r(x) \subset \mathbb{R}^2$ denote the closed disc with radius r and center x. Then A is called* simple *in x if there exists an $\varepsilon > 0$ such that every circle contained in $\mathcal{B}_\varepsilon(x)$ and surrounding x has the same finite number of intersecting points with ∂A (see Fig. 1). This number is noted as $n_A(x)$ and is called* local junction number *of A at position x. A is called* simple *if it is simple in every $x \in \mathbb{R}^2$.*

The digital reconstruction of an image is always simple due to the definition of pixels. Non-simple sets, like the set $\mathbb{Q}^2$ of points with rational coordinates, can obviously not be reconstructed topologically correctly. Thus it is not surprising that the local junction number does not change under $\mathbb{R}^2$-homeomorphisms for simple sets.

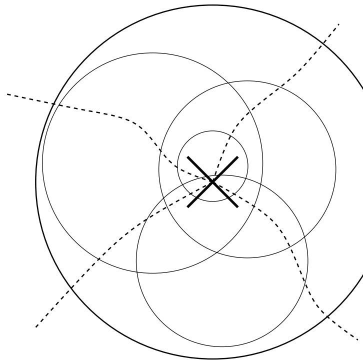


Figure 1. A set is simple if for each point there exists a radius ε, such that each circle surrounding x and having a smaller radius intersects the boundary of the set in the same number of points. This number is called local junction number.

LEMMA 3.2. *Let $A \subset \mathbb{R}^2$. The local junction number has the following properties:*

1. *$x \in A^0$ implies $n_A(x) = 0$.*
2. *$x \in (A^c)^0$ implies $n_A(x) = 0$.*
3. *Let f be an $\mathbb{R}^2$-homeomorphism with $B := f(A)$. Then $n_A(x) = n_B(f(x))$ if A is simple in x and B is simple in $f(x)$.*
4. *Let $\hat{A}$ be the digital reconstruction of A with some sampling grid S. Then $n_{\hat{A}}(x)$ is equal to zero or positive and even for every $x \in \mathbb{R}^2$.*
5. *Let A be a simple set and $x \in \mathbb{R}^2$ with $n_A(x) \geq 4$. Then for any ε there exists a circle of radius $r \leq \varepsilon$ with the following properties:*
   - *it surrounds x,*
   - *it intersects ∂A in exactly $n_A(x)$ points,*
   - *the angle between at least two of these intersection points relatively to the center of the circle is smaller than $\frac{\pi}{3}$.*
   - *the distance of its center to x is smaller than $\frac{\sqrt{3}-1}{2}\varepsilon$.*

Proof.

1. For any $x \in A^0$ there exists an ε such that $\mathcal{B}_\varepsilon(x) \subset A$.
2. For any $x \in (A^c)^0$ there exists an ε such that $\mathcal{B}_\varepsilon(x) \subset A^c$.
3. Since any $\mathbb{R}^2$-homeomorphism does not change the number of boundary lines meeting in x and $f(x)$, respectively, the local junction number of A in x has to be equal to the local junction number of B in $f(x)$.
4. Due to the definition of the digital reconstruction of some set, the local junction number is greater than two and even for any corner point, it is equal to two for any further boundary point and it is zero for the rest.
5. If $n_A(x) \geq 6$, then there exists a circle with center in x, which intersects ∂A in at least 6 points. Since $\frac{2\pi}{6} = \frac{\pi}{3}$, at least one pair of these points has an angle of at most $\frac{\pi}{3}$ relatively to x, due to the pidgeon hole principle. By a minimal displacement of the circle center we can enforce that this angle is smaller than $\frac{\pi}{3}$, i.e. 60 degrees.

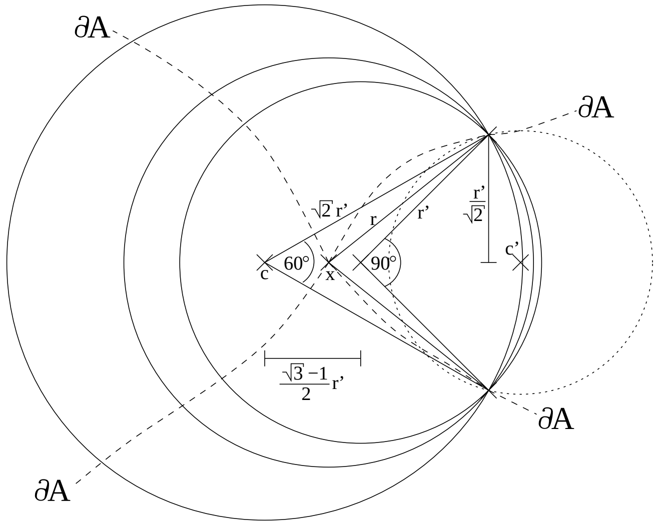


Figure 2. r lies in between r' and $\sqrt{2}r'$. Thus x is not more than $\frac{\sqrt{3}-1}{\sqrt{2}}r'$ from c away. In contrast c' has a distance of at least $\frac{1}{\sqrt{2}}r'$ to x.

Otherwise $n_A(x)$ has to be equal to 4. For similar reasons there exists a circle centered in x with two of four intersection points having an angle of at most $\frac{\pi}{2}$. W.l.o.g. let the radius r of the circle be smaller than $\frac{\varepsilon}{\sqrt{2}}$. Now there exists a unique circle going through the same two intersection points with a central angle of exactly $\frac{\pi}{3}$, which circumscribes x (see Fig. 2). As Fig. 2 illustrates, the radius of this circle is at most $\sqrt{2}r$ and its center has a distance to x of at most $\frac{\sqrt{3}-1}{\sqrt{2}}r$.

□

Obviously a simple set is a bordered 2D-manifold if and only if it has a local junction number of 2 in each boundary point. In case of non-manifold shapes the most interesting points are the ones with a local junction number of at least 4, since here they differ from manifold shapes. If one asks for a good digital reconstruction not only the topology should be preserved, but also the original and the reconstructed shapes should be as similar as possible, and especially any reconstructed junction point should be as near to the original one as possible. This leads to the following definition:

Definition 3.3. *A digital reconstruction $\hat{A}$ of a binary set A regarding to a sampling grid S is called* correct *if there exists an $\mathbb{R}^2$-homeomorphism mapping A to $\hat{A}$, such that each point with a local junction number greater than* 2 *is mapped to the nearest corner point in $\hat{A}$.*

A shape is called reconstructible *with some sampling grid, if any alignment and any downscaling of the grid (i.e. the grid becomes denser) leads to a correct digital reconstruction.*

Lemma 3.4. *If a non-manifold simple set is reconstructible with some sampling grid, each corner point of the grid has to be part of at least 7 pixels.*

Proof. A non-manifold simple set has at least one junction x with $n_A(x) \geq 4$. Due to Lemma 3.2.5 there exists a circle of radius r around x with the same number of intersection points with ∂A, where the angle between at least two such points is smaller than $\frac{\pi}{3}$. Now one can translate and downscale any grid such that the center c of the circle is a corner point of the grid and the sampling points whose pixels share this corner point c all lie on the circle. The intersection points cut the circle into sectors, such that each sector has to be hit by at least one sampling point in order to get the correct local junction number at the corner point. Since the grid can be arbitrarily rotated, the maximal angle α between two neighboring sampling points s_i, s_j at the circle has to be smaller than the smallest angle between two intersection points and thus smaller than $\frac{\pi}{3}$. Thus $6\alpha < 6\frac{\pi}{3} = 2\pi$, which implies that each corner point of the grid has to be part of at least 7 pixels.

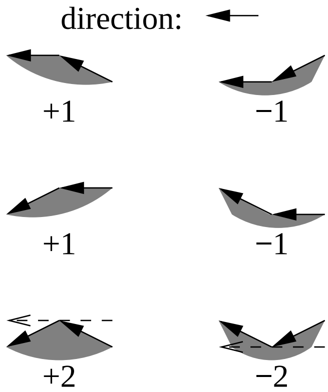


Figure 3. Regarding to a chosen direction, one only has to count the number of occurences of certain corner configurations with the given weights and divide the result by two in order to get the Euler characteristic. This algorithm can be applied to any Jordan polygon and thus to any shape preserving sampling grid.

It only remains to be shown that the corner point c is always the one being nearest to x. If $n_A(x) \geq 6$ this is obviously true. If otherwise $n_A(x) = 4$, the distance between c and x is bounded by $\frac{\sqrt{3}-1}{\sqrt{2}}r$. The only other corner point c', which is a candidate for being the nearest to x is the second one being part of the pixels of the two sampling points s_i, s_j (see Fig. 2). Since the radius of the circle centered in c' and going through s_i, s_j is at least $\frac{r'}{\sqrt{2}}$, this is also the minimal distance between x and c'. This distance is always greater than $\frac{\sqrt{3}-1}{\sqrt{2}}r'$ (see Fig. 2). □

Lemma 3.5. *There exists no r'-grid, such that each corner point is part of more than 6 pixels.*

Proof. The infinite graph G defined by the corner points as vertices and the pixel boundary lines as edges is obviously planar. If any corner point is part of at least 7 pixels, the minimal degree of G is 7. Thus if one chooses some center point x and looks at the degree of the subgraph which lies inside a disc with center in x, this degree converges to some value of at least 7 for increasing radius. In other words for any $\varepsilon > 0$ there exists an r such that the average degree of the subgraph lying inside the disc of radius r is at least $7 - \varepsilon$. This implies that the number of vertices of this subgraph is greater then $\frac{7}{2}$ times the number of edges (i.e. corner points). This is in contradiction to the well-known fact that for any planar graph the number of edges is at most as big as 3 times the number of vertices minus 6 (see Diestel[8]). Thus such a grid cannot exist. □

Theorem 3.6. *A simple set has to be a 2D-manifold in order to be reconstructible with some sampling grid.*

Proof. The theorem follows directly from Lemmas 3.4 and 3.5. □

4. IMPLICATIONS FOR DIGITAL IMAGES

In the last section It is shown that a shape cannot be guaranteed to be digitized correctly if it is no bordered 2D-manifold, regardless of the used grid structure.

So if you have a sampling grid S and a shape A, which remains similar under digitization with every alignment and downscaling of S, then not only A has to be a bordered 2D-manifold, but also the reconstruction $\hat{A}$. This implies that the local junction number of each point in the digital picture is 0 or 2.

For square grids these digital sets are the well-composed sets defined by Latecki.[5] Well-composed sets have the advantage that the definition of connectivity is straightforward: The components defined by edge connectivity are the same as the components defined by vertex connectivity. This is exactly the definition of the so-called *extended well-composed sets* for arbitrary sampling grids, which were introduced by Wang and Bhattacharya.[9]

Thus digital shapes, which are not extended well-composed, can not reliably tell something about the occurence of junctions in the original image. There are two reasons why junctions can occur in a digital reconstruction:

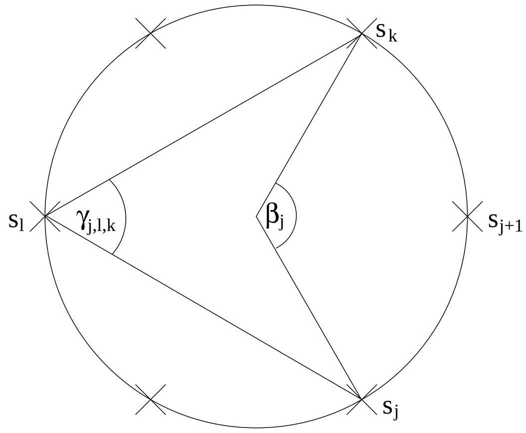


Figure 4. β_j is twice a big as $\gamma_{j,l,k}$. Due to the choice of s_j, s_k and s_l it follows that $\gamma_{j,l,k} \geq \frac{\pi}{3}$.

1. the used grid is unsuitable for digitizing the original shape (this also includes grids which are simply too coarse),

2. the original shape is no bordered 2D manifold – then another alignment and upscaling of the original shape relatively to the grid would cause a change in topology of the reconstruction.

Wang and Bhattacharya not only extended the definition of well-composed sets to arbitrary sampling grids, they also proved a Jordan curve theorem for extended well-composed sets[9]:

THEOREM 4.1. *The complement of a finite simple closed extended well-composed curve C has exactly two extended well-composed components and C is the common boundary of the two components.*

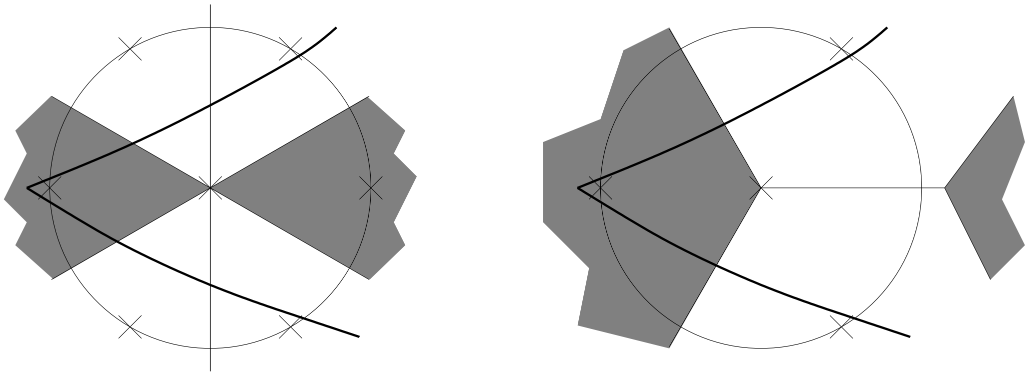
Since any extended well-composed set is a bordered 2D-manifold, extended well-composed sets have very interesting properties: Not only a digital Jordan curve theorem holds, but also the well-known connectivity paradoxes do not exist, because the components due to foreground- and background-connectivity do not differ. Moreover the Euler characteristic is locally computable, which is not the case in other digital images.

THEOREM 4.2. *In any extended well-composed set the Euler characteristic is locally computable.*

Proof. Any extended well-composed set is a bordered 2D-manifold. Thus its boundary components are Jordan curves. These curves have an intrinsic orientation, which is defined as follows: We simply follow any curve by keeping the set to our immediate left and the background to our immediate right. The sum of the direction changes at the corner points is 2π or minus 2π for each curve, regarding on its orientation, as already mentioned by Minsky and Papert.[10] If the curve is the outer boundary of a foreground component, then the sum is positive and otherwise it is negative. Thus the Euler characteristic, which denotes the number of foreground components minus the number of background components, is equal to the total sum of these sums divided by 2π and it is equal to the number of positively oriented curved minus the number of negatively oriented curves.

Now choose some direction δ arbitrarily. For any of these curves it is sufficient to do the following (see Fig. 3 for an illustration):

You have count the number of corners where the direction change is positive and one of the incident pixel edges has direction δ plus twice the number of corners where the direction change is positive, no incident pixel edge has direction δ and the line with direction δ going through the corner does not cut ∂A in every neighborhood of the corner (it only meets ∂A in the corner itself). Then you have to subtract the number of corners where the direction change is negative and one of the incident pixel edges has direction δ and you have to subtract twice the number of corners where the direction change is negative, no incident pixel edge has direction δ and the line with direction δ going through the corner does not cut ∂A in every neighborhood of the corner. Finally you have to divide the result by two and you get the Euler characteristic.



(a) (b)

Figure 5. If the boundary has corners with too small angles, some alignments of the sampling grid cause topological errors.

This algorithm bases on the following idea: If you have only differentiable Jordan curves as boundaries, where the direction is not constant for any boundary intervall, the Euler characteristic can be computed by counting the number of positively curved boundary points with a certain direction minus the number of negatively curved boundary points with the same direction, as shown by Lee, Poston and Rosenfeld.[11] Since we have polygons we simply count the number of boundary points where a boundary part having the chosen direction is achieved and the number of boundary points where such a boundary part is quitted. Boundary points where both happens have to be counted twice. Thus we count every occurence of the chosen directions twice and we have to divide the result by two. □

Note that the algorithm for computing the Euler characteristic for square grids mentioned by Lee et al.[11] can be seen as a special case of our algorithm where the direction is such chosen that no boundary line has the same normal direction and thus it is not necessary to count each occurence twice and divide the result by two.

5. RESTRICTIONS FOR BOUNDARY ANGLES

In section 3 it was proven hat a shape has to be a bordered 2D-manifold in order to be digitized correctly with any alignment and downscaling of the sampling grid. Although this is a very hard restriction, not every bordered 2D-manifold can be digitized correctly with any type of sampling grid. E.g. if the boundary of a shape has a corner with a very small angle, the topology can change during digitization as illustrated in Fig. 5. Obviously the size of the angle where this problem begins to occur heavily depends on the structure of the sampling grid. Nevertheless it is possible to prove a lower bound which is true for every sampling grid:

THEOREM 5.1. *Let $A \subset \mathbb{R}^2$ be a shape, such that its boundary ∂A has an angle smaller than $\frac{\pi}{3}$ in some $x \in \partial A$. Then A is not reconstructible with any sampling grid. If the sampling grid has corner points which are part of exactly* 5 *(*4*) pixels, even sets with boundary angles smaller than $\frac{2}{5}\pi$ ($\frac{\pi}{2}$) are not reconstructible.*

Proof. Due to Lemma 3.5 there exists at least one corner point c being part of at most 6 pixels. The corresponding up to 6 sampling points lie on a common circle with center c.

Suppose there are 6 such sampling points, which are noted in clockwise order as $s_1, s_2, s_3, s_4, s_5, s_6$. Now let $\alpha_i := \angle(s_i, c, s_{i+1})$ with $s_7 := s_1$ and let $\beta_j = \alpha_j + \alpha_{j+1}$ with $\alpha_7 := \alpha_1$. Since the sum of all β_j ($j \in \{1, 2, 3, 4, 5, 6\}$) is equal to $2 \cdot 2\pi$, at least one β_j must be at least $\frac{2 \cdot 2\pi}{6} = 2\frac{\pi}{3}$. Now let s_j, s_k be the two sampling points, which correspond to β_j. Due to simple geometry the central angle $\beta_j = \angle(s_j, c, s_k)$ is exactly twice any inscribed angle $\gamma_{j,l,k} = \angle(s_j, s_l, s_k)$ with $s_l \notin \{s_j, s_{j+1}, s_k\}$ being one of the remaining 3 sampling points (see Fig. 4). Thus $\gamma_{j,l,k} \geq \frac{\pi}{3}$. Since the boundary angle of A at x is smaller than $\frac{\pi}{3}$, one can place a sufficiently downscaled version of the sampling grid such that an additional junction occurs in the digital reconstruction, as shown in Fig. 5(a). Analogously an angle smaller than $\frac{2}{5}\pi$ is not allowed if there exists a corner point c being part of exactly 5 pixels, and an angle smaller than $\frac{\pi}{2}$ is not allowed if there exists a corner point c being part of exactly

4 pixels. The final remaining case are sampling grids where each corner point is part of only 3 pixels. Then let c be such a corner point and s_1, s_2, s_3 be the three sampling points. Then at least one of the three angles $\angle(s_1, s_2, s_3), \angle(s_2, s_1, s_3), \angle(s_1, s_3, s_2)$ is at least $\frac{\pi}{3}$. Then, similarily to the other cases, since the boundary angle of A at x is smaller than $\frac{\pi}{3}$, one can place a sufficiently downscaled version of the sampling grid such that an additional component occurs in the digital reconstruction, as shown in Fig. 5(b).

Thus angles smaller than $\frac{\pi}{3}$ in the boundary imply a set not to be reconstructible with any sampling grid. □

In case of the commonly used square grid this means that any acute angle can lead to a topologically incorrect digital reconstruction. Interestingly this is the worst case of all sampling grids. E.g. in case of regular sampling grids not only the hexagonal grid – which is known for its nice topological behaviour – but also the triangular grid are better candidates to digitize corners topologically correctly.

The proofs in this paper make use of the missing of any restriction of the density of the sampling grid. This shows that supersampling is not always good and that the intuitive assumption that a denser sampling grid would lead to a better digitization (in a topological sense) is simply wrong.

6. CONCLUSIONS

It is proven that given any sampling grid a binary image has to be a bordered 2D-manifold if one wants to guarantee that the digital reconstruction with some arbitrarily dense and aligned version of the sampling grid is correct in a topological sense. This has the far reaching implication for the digital reconstruction, that it is an extended well-composed set and thus a digital version of the Jordan curve theorem holds and the Euler characteristic is locally computable. This allows a lot of image analysis algorithms to be more effective and – which is even more interesting – the well-known connectivity paradoxons do not occur. Thus every time one has to deal with such a paradoxon, it is due to a non-reconstructible shape or due to a bad sampling grid.

Moreover it is proven that the boundary of a binary image has to have no angles smaller than $60°$ regardless of the sampling grid type. In case of square grids even any acute angle can lead to a topologically incorrect digital reconstruction.

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