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Obstacles on the Way to Spatial Reasoning with Description Logics: Undecidability of $\mathcal{ALC}_{\mathcal{RA}^{\ominus}}$

(Slightly Revised Version)

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Abstract

This paper presents the new description logic $\mathcal{ALC}_{\mathcal{RA}^\ominus}$. $\mathcal{ALC}_{\mathcal{RA}^\ominus}$ combines the well-known standard description logic $\mathcal{ALC}$ with composition-based role axioms of the form $S \circ T \sqsubseteq R_1 \sqcup \cdots \sqcup R_n$. We argue that these axioms are nearly indispensable components in a description logic framework suitable for qualitative spatial reasoning tasks. An $\mathcal{ALC}_{\mathcal{RA}^\ominus}$ spatial reasoning example is presented, and the relationships to other descriptions logics are discussed (namely $\mathcal{ALC}_{\mathcal{RA}}$, $\mathcal{ALC}_{\mathcal{R}^+}$, $\mathcal{ALC}_\oplus$, $\mathcal{ALCH}_{\mathcal{R}^+}$). Unfortunately, the satisfiability problem of this new logic is undecidable. Due to the high relevance of role axioms of the proposed form for all kinds of qualitative reasoning tasks, the undecidability of $\mathcal{ALC}_{\mathcal{RA}^\ominus}$ is an important result.

1 Introduction and Motivation

Since the introduction of KL-ONE (see [2]), knowledge representation systems based on description logics (DLs) have been proven valuable tools in the field of formal knowledge representation. Description logic systems offer formally defined syntax and semantics, which enables the unambiguous specification of the services offered to users of these systems. In fact, many early knowledge representation systems and frameworks suffered from unclear semantics (e.g. see [23] for an overview and discussion). In many cases the underlying base description logic of a DL-based system can be seen as a subset of first order predicate logic (FOPL). In contrast to FOPL, decidability of diverse inference problems is usually guaranteed for description logics, for example, for the *satisfiability problem* of formulas. Please recall that the satisfiability problem is only semi-decidable for FOPL. Moreover, for (less expressive) description logics even tractable (deterministic polynomial-time) inference algorithms have been found (see [4, 5]). The merits of description logics are widely recognized, and a remarkable amount of research covering theory and practice has been carried out during the last 20 years. However, mediation between expressiveness and tractability remained a problem.

Description logics focus on the structural description of unary and binary predicates. Unary predicates are called *concepts*, and binary predicates correspond to so-called *roles*. Sometimes DLs are even called *concept description languages* – indicating that the focus has traditionally been more on the side of the concept descriptions than on the side of the role descriptions. In our opinion the ability to interrelate roles via some kind of constraints has not been investigated as thoroughly as the concept description side of DLs. For example, see [3] for a description logic providing *role conjunction*. In contrast to role conjunction, *role disjunction* is not interesting in most description logics. *Role negation* has only been considered very recently. Role inclusion axioms have also been considered. However, the space of possibilities for role axioms resp. formulas relating roles to one another has not been exhaustively examined. To the best of our knowledge, the concept satisfiability problem w.r.t. to a set of role axioms of the proposed form has not been considered before.

Like for formulas in FOPL, the syntax of the concepts is determined by a set of concept forming operators and a set of atomic components, so-called role names and concept names. The semantics of the syntactic elements is then specified by giving a Tarski-style *interpretation*. An interpretation maps concepts and roles to unary resp. binary relations on the non-empty interpretation domain: concepts are therefore mapped to subsets of the interpretation domain, and roles to sets of tuples of domain objects. The denoted (unary or binary) relation is also called the *extension* of the concept or role.

If the semantics of the operators is preserved by the mapping and the extension of a concept is non-empty, then the interpretation is said to be a *model* of that concept. Given an arbitrary concept C of the language, the most important inference problem is to decide whether C has a model. In this case, C is called *satisfiable.*

Before we discuss the modeling of *spatial* concepts, let us consider some non-spatial concepts. For example, the unary FOPL predicate $father\_with\_son(x)$ could be defined by means of the FOPL formula $human(x) \land male(x) \land \exists y : has\_child(x,y) \land human(y) \land male(y)$. Here, *human* and *male* are unary predicate names, whereas *has_child* is a binary predicate name. Translated into the variable-free description logic syntax we would get $human \sqcap male \sqcap$

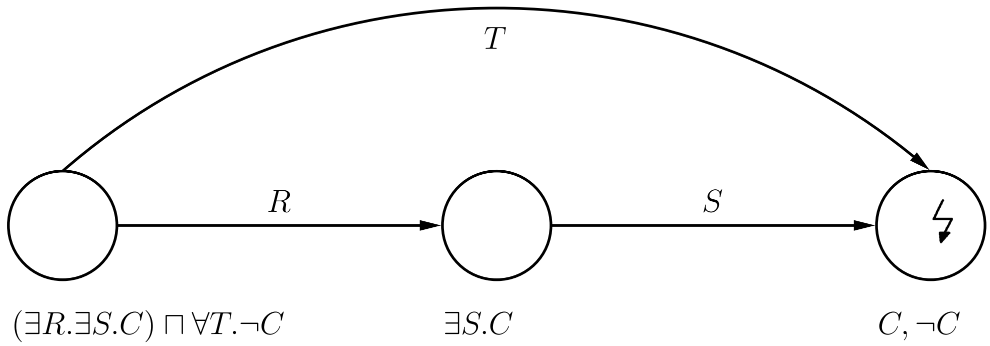

Figure 1: Simple Example

$\exists$*has_child*.(*human* $\sqcap$ *male*). The whole expression is a concept, *human* and *male* are concept *names* or *atomic* concepts, *has_child* is a role (name), and $\sqcap$ and $\exists$ are concept-forming operators.

Obviously, if roles are not related to one another by some kind of constraints, we cannot claim to have represented inherent properties that would be natural for some relationships. For example, in order to appropriately capture the meaning of the relationship "niece", one would have to ensure that a brother's or a sister's daughter is indeed a niece of this very same person. In FOPL this requirement could be expressed by means of two (conjunctively combined) universally quantified statements of the form $\forall x, y, z : (has\_brother(x, y) \wedge has\_daughter(y, z) \Rightarrow has\_niece(x, z))$ and $\forall x, y, z : (has\_sister(x, y) \wedge has\_daughter(y, z) \Rightarrow has\_niece(x, z))$. The interpretation of the role *has_niece* is then no longer independent from the interpretations of the roles *has_brother* (resp. *has_sister*) and *has_daughter*. The role axioms of $\mathcal{ALC}_{\mathcal{RA}^\ominus}$ allow to express global universally quantified implication statements exactly like these: in fact, these formulas are equivalent to the $\mathcal{ALC}_{\mathcal{RA}^\ominus}$ role axioms $has\_brother \circ has\_daughter \sqsubseteq has\_niece$ and $has\_sister \circ has\_daughter \sqsubseteq has\_niece$.

In the following we assume that the reader is familiar with description logics, at least with the basic logic $\mathcal{ALC}$ (see [20] and [23] for an introduction). Basically $\mathcal{ALC}_{\mathcal{RA}^\ominus}$ augments the standard description logic $\mathcal{ALC}$ with composition-based role axioms of the form $S \circ T \sqsubseteq R_1 \sqcup \cdots \sqcup R_n$, $n \geq 1$, enforcing $S^\mathcal{I} \circ T^\mathcal{I} \subseteq R_1^\mathcal{I} \cup \cdots \cup R_n^\mathcal{I}$ on the models $\mathcal{I}$. This corresponds to a universally quantified FOPL formula of the form $\forall x, y, z : (S(x, y) \wedge T(y, z) \Rightarrow R_1(x, z) \vee \cdots \vee R_n(x, z))$. A finite set of these role axioms is called a *role box* and is denoted by $\mathfrak{R}$.

Please consider the $\mathcal{ALC}$ concept $(\exists R.\exists S.C) \sqcap \forall T.\neg C$; in FOPL:
$(\exists[x]((\exists[y](R(x, y) \wedge \exists[x](S(y, x) \wedge C(x)))) \wedge (\forall[y](T(x, y) \Rightarrow \neg C(y)))))$.

Obviously, this concept is satisfiable, since the FOPL formula is satisfiable. However, the same concept is unsatisfiable in $\mathcal{ALC}_{\mathcal{RA}^\ominus}$ w.r.t. the role box

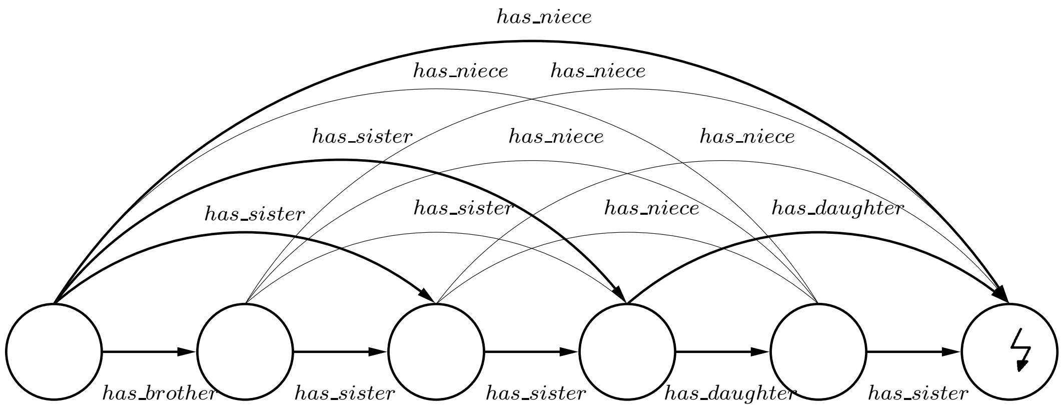

Figure 2: Complex Example

$\{R \circ S \sqsubseteq T\}$. Again, translated into FOPL we have
$(\forall[x,y,z](R(x,y) \land S(y,z) \Rightarrow T(x,z))) \land$
$(\exists[x]((\exists[y](R(x,y) \land \exists[x](S(y,x) \land C(x)))) \land (\forall[y](T(x,y) \Rightarrow \neg C(y))))).$

The only difference to the former formula is the additional conjunct in the first line, expressing the role axiom. In fact, $(\forall[x,y,z](R(x,y) \land S(y,z) \Rightarrow T(x,z)))$ enforces the presence of $T(x,z)$ because of $\exists[x]((\exists[y](R(x,y) \land \exists[x](S(y,x) \land C(x)))) \ldots)$, and then the qualification $\forall[y](T(x,y) \Rightarrow \neg C(y))$ is applicable, yielding an inconsistency since also C holds for this individual.

As another example taken from the realm of genealogy, let us consider the concept expression

$$(\exists \mathit{has\_brother}.\exists \mathit{has\_sister}.\exists \mathit{has\_sister}.\exists \mathit{has\_daughter}.\exists \mathit{has\_sister}.\\ \mathit{computer\_science\_student}) \sqcap \forall \mathit{has\_niece}.\neg \mathit{computer\_science\_student}$$

w.r.t. the role box

$$\begin{array}{lllll} \{ & \mathit{has\_brother} & \circ\ \mathit{has\_sister} & \sqsubseteq\ \mathit{has\_sister}, & \\ & \mathit{has\_sister} & \circ\ \mathit{has\_daughter} & \sqsubseteq\ \mathit{has\_niece}, & \\ & \mathit{has\_daughter} & \circ\ \mathit{has\_sister} & \sqsubseteq\ \mathit{has\_daughter}, & \\ & \mathit{has\_sister} & \circ\ \mathit{has\_sister} & \sqsubseteq\ \mathit{has\_sister} & \}. \end{array}$$

A careful inspection reveals that this concept is inconsistent w.r.t. this role box, since the computer science student plays also the role of a niece and is therefore a filler of the *has_niece* role, see Figure 2.

Note that composition of roles is not allowed to appear on the right hand side of role axioms. One can therefore not write axioms like $\mathit{has\_niece} \sqsubseteq (\mathit{has\_brother} \circ \mathit{has\_daughter}) \sqcup (\mathit{has\_sister} \circ \mathit{has\_daughter})$. The rationale for this restriction is that it is known since 1989 that allowing composition also on the right hand side of role axioms would yield a form of undecidability that is also present in the so-called *role value maps* (see [19]). For the same reason, $\mathcal{ALC}_{\mathcal{RA}^\ominus}$ does not include *inverse* roles. As we show in this paper, it suffices to allow composition on the left-hand side to make the resulting logic undecidable. This is a new and unexpected result. The proof techniques applied in [19] to show the undecidability of role value maps cannot be exploited to show the undecidability of $\mathcal{ALC}_{\mathcal{RA}^\ominus}$, because the proof given in [19] strongly depends on the presence of role compositions on the *right* hand side of implication axioms.

As discussed below in the spatial reasoning example, axioms of the form $S \circ T \sqsubseteq R_1 \sqcup \cdots \sqcup R_n$ seem to be indispensable components in a description logic framework suitable for qualitative spatial reasoning tasks. The discovered undecidability result is therefore a big obstacle on the way to a full-fledged spatial-reasoning description-logic framework which would even need *more* expressiveness than provided by $\mathcal{ALC}_{\mathcal{RA}^\ominus}$. For example, in order to truly capture the semantics of qualitative spatial relationships like the ones discussed below, *inverse* roles and additional role disjointness declarations would be needed.

In [22], we presented the logic $\mathcal{ALC}_{\mathcal{RA}}$. The only difference between $\mathcal{ALC}_{\mathcal{RA}}$ and $\mathcal{ALC}_{\mathcal{RA}^\ominus}$ is that the former requires that all roles are interpreted as disjoint, i.e. for any two roles R, S with $R \neq S$ and any interpretation $\mathcal{I}$, $R^\mathcal{I} \cap S^\mathcal{I} = \emptyset$ must hold. Even though this seems to be a minor variation of $\mathcal{ALC}_{\mathcal{RA}^\ominus}$, in fact it is not, because the disjointness requirement for roles has a number of non-obvious and far-reaching consequences (see [22]). The undecidability proof given here does not apply to $\mathcal{ALC}_{\mathcal{RA}}$, so the question whether $\mathcal{ALC}_{\mathcal{RA}}$ is decidable or not is still open.

The structure of this paper is as follows: first we will formally define the syntax and semantics of $\mathcal{ALC}_{\mathcal{RA}^\ominus}$. Then, the relationships to other known description logics providing some kind of transitive roles are sketched. The usefulness of $\mathcal{ALC}_{\mathcal{RA}^\ominus}$ in a spatial reasoning scenario is exemplified in the next section. The main contribution of this paper is the undecidability proof in Section 5. Finally, we conclude by discussing whether $\mathcal{ALC}_{\mathcal{RA}}$ might be undecidable as well, and future work is outlined. In the search of a decidable description logic with composition-based role axioms of the proposed form, a promising idea is to impose certain syntactic restrictions on the allowed role boxes. These syntactic restrictions have to be worked out in the future.

2 Syntax and Semantics of $\mathcal{ALC}_{\mathcal{RA}^\ominus}$

In the following the set of well-formed concepts of $\mathcal{ALC}_{\mathcal{RA}^\ominus}$ is specified:

Definition 1 (Concept Expressions) Let $\mathcal{N_C}$ be a set of concept names, and let $\mathcal{N_R}$ be a set of role names (roles for short), such that $\mathcal{N_C} \cap \mathcal{N_R} = \emptyset$. The set of concept expressions (or concepts for short) is the smallest inductively defined set such that

1. Every concept name $C \in \mathcal{N_C}$ is a concept.

2. If C and D are concepts, and $R \in \mathcal{N_R}$ is a role, then the following expressions are concepts as well: $(\neg C)$, $(C \sqcap D)$, $(C \sqcup D)$, $(\exists R.C)$, and $(\forall R.C)$.

3. Nothing else is a concept. ■

The set of concepts is the same as for the language $\mathcal{ALC}$. If a concept starts with "(", we call it a compound concept, otherwise a concept name or atomic concept. Brackets may be omitted for the sake of readability if the concept is still uniquely parsable.

We use the following abbreviations: if $R_1, \ldots, R_n$ are roles, and C is a concept, then we define $(\forall R_1 \sqcup \cdots \sqcup R_n.C) =_{def} (\forall R_1.C) \sqcap \cdots \sqcap (\forall R_n.C)$ and $\exists R_1 \sqcup \cdots \sqcup R_n.C =_{def} (\exists R_1.C) \sqcup \cdots \sqcup (\exists R_n.C)$. Additionally, for some $CN \in \mathcal{N_C}$ we define $\top =_{def} CN \sqcup \neg CN$ and $\bot =_{def} CN \sqcap \neg CN$ (therefore, $\top^\mathcal{I} = \Delta^\mathcal{I}$, $\bot^\mathcal{I} = \emptyset$).

The set of *roles* being used within a concept term C is defined:

Definition 2 (Used Roles, $\mathsf{roles}(C)$)

$$\mathsf{roles}(C) =_{def} \begin{cases} \emptyset & \text{if } C \in \mathcal{N_C} \\ \mathsf{roles}(D) & \text{if } C = (\neg D) \\ \mathsf{roles}(D) \cup \mathsf{roles}(E) & \text{if } C = (D \sqcap E) \\ & \text{or } C = (D \sqcup E) \\ \{R\} \cup \mathsf{roles}(D) & \text{if } C = (\exists R.D) \\ & \text{or } C = (\forall R.D) \end{cases}$$ ■

For example, $\mathsf{roles}(\forall R.\exists.SC \sqcap \exists T.D) = \{R, S, T\}$.

As already noted, $\mathcal{ALC}_{\mathcal{RA}^\ominus}$ provides role axioms of the form $S \circ T \sqsubseteq R_1 \sqcup \cdots \sqcup R_n$. More formally, the syntax of these role axioms is as follows:

Definition 3 (Role Axioms, Role Box) If $S, T, R_1, \ldots, R_n \in \mathcal{N_R}$, then the expression $S \circ T \sqsubseteq R_1 \sqcup \cdots \sqcup R_n$, $n \geq 1$, is called a *role axiom*. If $ra = S \circ T \sqsubseteq R_1 \sqcup \cdots \sqcup R_n$, then $\mathsf{pre}(ra) =_{def} (S, T)$ and $\mathsf{con}(ra) =_{def} \{R_1, \ldots, R_n\}$. A finite set $\mathfrak{R}$ of role axioms is called a *role box*. Let $\mathsf{roles}(ra) =_{def} \{S, T, R_1, \ldots, R_n\}$, and $\mathsf{roles}(\mathfrak{R}) =_{def} \bigcup_{ra \in \mathfrak{R}} \mathsf{roles}(ra)$. □

Additionally, a set of global *concept inclusion axioms (GCIs)* can be specified. A set of these GCIs is called a free TBox:

Definition 4 (Generalized Concept Inclusion Axiom, TBox) If C and D are $\mathcal{ALC}_{\mathcal{RA}^\ominus}$ concepts, then the expression $C \dot{\sqsubseteq} D$ is called a *generalized concept inclusion axiom,* or GCI for short. A finite set of such GCIs is called a free *TBox*, $\mathfrak{T}$. We use $C \doteq D \in \mathfrak{T}$ as a shorthand for $\{C \dot{\sqsubseteq} D, D \dot{\sqsubseteq} C\} \subseteq \mathfrak{T}$. □

The semantics of an $\mathcal{ALC}_{\mathcal{RA}^\ominus}$ concept is specified by giving a Tarski-style interpretation $\mathcal{I}$ that has to satisfy the following conditions:

Definition 5 (Interpretation) An interpretation $\mathcal{I} =_{def} (\Delta^\mathcal{I}, \cdot^\mathcal{I})$ consists of a non-empty set $\Delta^\mathcal{I}$, called the domain of $\mathcal{I}$, and an interpretation function $\cdot^\mathcal{I}$ that maps every concept name to a subset of $\Delta^\mathcal{I}$, and every role name to a subset of $\Delta^\mathcal{I} \times \Delta^\mathcal{I}$.

The interpretation function $\cdot^\mathcal{I}$ can then be extended to arbitrary concepts C by using the following definitions (we write $X^\mathcal{I}$ instead of $\cdot^\mathcal{I}(X)$):

$$
\begin{array}{rcl}
(\neg C)^\mathcal{I} & =_{def} & \Delta^\mathcal{I} \setminus C^\mathcal{I} \\
(C \sqcap D)^\mathcal{I} & =_{def} & C^\mathcal{I} \cap D^\mathcal{I} \\
(C \sqcup D)^\mathcal{I} & =_{def} & C^\mathcal{I} \cup D^\mathcal{I} \\
(\exists R.C)^\mathcal{I} & =_{def} & \{\, i \in \Delta^\mathcal{I} \mid \exists j \in C^\mathcal{I} : \lessdot i, j \gtrdot \in R^\mathcal{I} \,\} \\
(\forall R.C)^\mathcal{I} & =_{def} & \{\, i \in \Delta^\mathcal{I} \mid \forall j : \lessdot i, j \gtrdot \in R^\mathcal{I} \Rightarrow j \in C^\mathcal{I} \,\}
\end{array}
$$
□

It is therefore sufficient to provide the interpretations for the concept *names* and the roles, since the interpretation of every concept is uniquely determined then by using the definitions.

In the following we specify under which conditions a given interpretation is a *model* of a syntactic entity (we also say an interpretation *satisfies* a syntactic entity):

Definition 6 (Model Relationship) An interpretation $\mathcal{I}$ satisfies / is a model of a concept C, written $\mathcal{I} \models C$, iff $C^\mathcal{I} \neq \emptyset$.

An interpretation $\mathcal{I}$ satisfies / is a model of a role axiom $S \circ T \sqsubseteq R_1 \sqcup \cdots \sqcup R_n$, written $\mathcal{I} \models S \circ T \sqsubseteq R_1 \sqcup \cdots \sqcup R_n$, iff $S^\mathcal{I} \circ T^\mathcal{I} \subseteq R_1^\mathcal{I} \cup \cdots \cup R_n^\mathcal{I}$.

An interpretation $\mathcal{I}$ satisfies / is a model of a role box $\mathfrak{R}$, written $\mathcal{I} \models \mathfrak{R}$, iff for all role axioms $ra \in \mathfrak{R}$: $\mathcal{I} \models ra$.

An interpretation $\mathcal{I}$ satisfies / is a model of a GCI $C \dot{\sqsubseteq} D$, written $\mathcal{I} \models C \dot{\sqsubseteq} D$, iff $C^\mathcal{I} \subseteq D^\mathcal{I}$.

An interpretation $\mathcal{I}$ satisfies / is a model of a TBox $\mathfrak{T}$, written $\mathcal{I} \models \mathfrak{T}$, iff for all GCIs $g \in \mathfrak{T}$: $\mathcal{I} \models g$.

An interpretation $\mathcal{I}$ satisfies / is a model of $(C, \mathfrak{R})$, written $\mathcal{I} \models (C, \mathfrak{R})$, iff $\mathcal{I} \models C$ and $\mathcal{I} \models \mathfrak{R}$.

An interpretation $\mathcal{I}$ satisfies / is a model of $(C, \mathfrak{R}, \mathfrak{T})$, written $\mathcal{I} \models (C, \mathfrak{R}, \mathfrak{T})$, iff $\mathcal{I} \models C$, $\mathcal{I} \models \mathfrak{R}$ and $\mathcal{I} \models \mathfrak{T}$. □

Definition 7 (Satisfiability) A syntactic entity (concept, role box, concept with role box, etc.) is called *satisfiable* iff there is an interpretation which satisfies this entity; i.e. the entity has a model. ■

Then, the *satisfiability problem* is to decide whether a syntactic entity is satisfiable or not.

An important relationship between concepts is the subsumption relationship, which is a partial ordering on concepts w.r.t. their specificity:

Definition 8 (Subsumption Relationship) A concept D *subsumes* a concept C, $C \sqsubseteq D$, iff $C^{\mathcal{I}} \subseteq D^{\mathcal{I}}$ holds for all interpretations $\mathcal{I}$. □

Since $\mathcal{ALC}_{\mathcal{RA}^\ominus}$ provides a full negation operator, the subsumption problem can be reduced to the concept satisfiability problem: $C \sqsubseteq D$ iff $C \sqcap \neg D$ is unsatisfiable.

It should be noticed that a satisfiability tester for $\mathcal{ALC}_{\mathcal{RA}^\ominus}$ would also be able to determine satisfiability resp. subsumption w.r.t. free TBoxes. Each concept inclusion axiom can be dealt with by a technique called *internalization* (see [13, 14, 1]). Internalization for $\mathcal{ALC}_{\mathcal{RA}^\ominus}$ works as follows. Let $(C, \mathfrak{R}, \mathfrak{T})$ be the concept, role box and free TBox to be tested for satisfiability. Let $R_? \in \mathcal{N_R}$ be some role such that $R_? \notin \mathsf{roles}(C) \cup \mathsf{roles}(\mathfrak{R})$. Referring to $R_?$ and $(C, \mathfrak{R}, \mathfrak{T})$, the role box $\mathfrak{R}$ is completed: $\mathfrak{R}' = \mathfrak{R} \cup \{ R \circ S \sqsubseteq R_? \mid R, S \in (\{R_?\} \cup \mathsf{roles}(\mathfrak{R})),$
$\neg \exists ra \in \mathfrak{R} : \mathsf{pre}(ra) = (R, S) \}$.

Now, $(C, \mathfrak{R}, \mathfrak{T})$ is satisfiable iff $((C \sqcap M_{\mathfrak{T}} \sqcap \forall * .M_{\mathfrak{T}}), \mathfrak{R}')$ is satisfiable, where $\forall * .M_{\mathfrak{T}}$ is an abbreviation for $\forall (\sqcup_{R \in \mathsf{roles}(\mathfrak{R}')} R).M_{\mathfrak{T}}$. $M_{\mathfrak{T}}$ is the so-called *meta-constraint* corresponding to the TBox $\mathfrak{T}$: $M_{\mathfrak{T}} =_{def} \sqcap_{C \dot{\sqsubseteq} D \in \mathfrak{T}} (\neg C \sqcup D)$.

3 Relationships to Other Logics

In order to judge the expressive power of $\mathcal{ALC}_{\mathcal{RA}^\ominus}$ we consider other logics and examine whether they are subsumed[1] by $\mathcal{ALC}_{\mathcal{RA}^\ominus}$. We briefly sketch the relationships to the most important base description logics offering some form of transitivity. Additionally, (un)decidability results regarding transitivity extensions of the so-called (loosely) guarded fragment of FOPL are briefly discussed

[1] We say that a language A is "subsumed" by a language B (resp. provides the same expressive power) iff the satisfiability problem of A can be reduced to the satisfiability problem of B.

in order to check whether they apply to $\mathcal{ALC}_{\mathcal{RA}^\ominus}$. The discussion provides some key-insights into the high expressiveness of composition-based role axioms.

$\mathcal{ALC}_{\mathcal{R}^+}$: The description logic $\mathcal{ALC}_{\mathcal{R}^+}$ augments $\mathcal{ALC}$ (see [17]) with transitively closed roles. A role R may be declared as transitively closed which enforces (for every model $\mathcal{I}$) $R^\mathcal{I} = (R^\mathcal{I})^+$. $\mathcal{ALC}_{\mathcal{R}^+}$ is obviously a proper sub-fragment of $\mathcal{ALC}_{\mathcal{RA}^\ominus}$, since a role R can be declared as transitively closed with the role axiom $R \circ R \sqsubseteq R$, which enforces $(R^\mathcal{I})^+ \subseteq R^\mathcal{I}$ and therefore $R^\mathcal{I} = (R^\mathcal{I})^+$. The concept satisfiability problem of $\mathcal{ALC}_{\mathcal{R}^+}$ is decidable and PSPACE-complete. $\mathcal{ALC}_{\mathcal{R}^+}$ is basically just a syntactic variant of the multi-modal logic $K4_n$, with n transitive accessibility relations; plain $\mathcal{ALC}$ corresponds to K_n (see [18]). The n accessibility relations correspond to n different roles. The only difference between $\mathcal{ALC}_{\mathcal{R}^+}$ and $K4_n$ is that the latter requires that *all* n accessibility relations are transitively closed, whereas the transitive closure of a role is optionally in $\mathcal{ALC}_{\mathcal{R}^+}$.

$\mathcal{ALC}_+$ and $\mathcal{ALC}_\oplus$: As Sattler points out, $\mathcal{ALC}_{\mathcal{R}^+}$ is not capable to distinguish "direct" and "indirect" successors of a transitively closed role R. Baader had already introduced the language $\mathcal{ALC}_+$ (see [1]) which provides a transitive closure *operator* (in fact, $\mathcal{ALC}_+$ is more or less a notational variant of the *Propositional Dynamic Logic, PDL*, see [18]). Both a "generating" role R and its transitive closure $+(R)$ can be distinguished and used separately within concepts. $(+(R))^\mathcal{I} = (R^\mathcal{I})^+$ is enforced. $\mathcal{ALC}_+$ is no longer a subset of FOPL, since the transitive closure of a role cannot be expressed in FOPL (this violates the compactness of FOPL). However, it can be expressed in FOPL that a role R is transitively closed: $\forall x, y, z : R(x, y) \wedge R(y, z) \Rightarrow R(x, z)$. There is no way to simulate the expressiveness of $\mathcal{ALC}_+$ in $\mathcal{ALC}_{\mathcal{RA}^\ominus}$, since the latter is still a subset of FOPL, but the former is not. The concept satisfiability problem of $\mathcal{ALC}_+$ is decidable and EXPTIME-complete. In the search for a computationally less expensive logic, Sattler introduced the language $\mathcal{ALC}_\oplus$ (see [17]), which replaces the transitive closure operator "+" with the so-called "transitive orbit" operator "$\oplus$". Like the "+"-operator, the transitive orbit operator can be applied to roles. Applied to a role R, the role $\oplus(R)$ is interpreted as *some* relation being a superset of the transitive closure of the generating role R, but not necessarily the smallest one: only $(R^\mathcal{I})^+ \subseteq (\oplus(R))^\mathcal{I}$ is granted. The concept satisfiability problem of $\mathcal{ALC}_\oplus$ is decidable but unfortunately, as Sattler has shown, EXPTIME-complete (as for $\mathcal{ALC}_+$, too).

We show that $\mathcal{ALC}_\oplus$ is subsumed by $\mathcal{ALC}_{\mathcal{RA}^\ominus}$ by reducing the concept satisfiability problem of $\mathcal{ALC}_\oplus$ to the concept satisfiability problem of $\mathcal{ALC}_{\mathcal{RA}^\ominus}$: given an $\mathcal{ALC}_\oplus$ concept C, we construct a concept C' and a role box $\mathfrak{R}'$ such that

$(C', \mathfrak{R}')$ is satisfiable in $\mathcal{ALC}_{\mathcal{RA}^\ominus}$ iff the original concept C is.[2]

C' is constructed from C as follows: The role $\oplus(R)$ in C is replaced by the role $R_\oplus$. Then, for every role $R_\oplus$, we add the role axioms $\{R \circ R \sqsubseteq R_\oplus, R_\oplus \circ R \sqsubseteq R_\oplus\}$ to $\mathfrak{R}$. Please note that this only ensures $(\oplus(R))^\mathcal{I} = R^\mathcal{I} \cup R_\oplus^\mathcal{I}$, and *not* $(\oplus(R))^\mathcal{I} = R_\oplus^\mathcal{I}$, since $R^\mathcal{I} \not\subseteq R_\oplus^\mathcal{I}$. Therefore, in order to get an equi-satisfiable concept C', we have to rewrite the original concept C in the following way:

$$\begin{aligned} \exists \oplus (R).D &\rightarrow \exists R_\oplus.D \\ \exists R.D &\rightarrow \exists R_\oplus.D \sqcap \exists R.D \\ \forall \oplus (R).D &\rightarrow \forall R_\oplus.D \sqcap \forall R.D \end{aligned}$$

Now, C' is satisfiable w.r.t. the role box $\mathfrak{R}$ iff C is satisfiable.

$\mathcal{ALCH}_{\mathcal{R}^+}$: The description logic $\mathcal{ALCH}_{\mathcal{R}^+}$ (see [13, 14]) extends $\mathcal{ALC}_{\mathcal{R}^+}$ by an additional set of role inclusion axioms of the form $R \sqsubseteq S$, enforcing $R^\mathcal{I} \subseteq S^\mathcal{I}$ on the models $\mathcal{I}$. Adding the identity role Id with the fixed semantics of the identity relationship $Id^\mathcal{I} =_{def} \{ <x, x> \mid x \in \Delta^\mathcal{I} \}$ to $\mathcal{ALC}_{\mathcal{RA}^\ominus}$ would obviously enable the simulation of these role inclusion axioms: for each role inclusion axiom $R \sqsubseteq S$, add the role axiom $R \circ Id \sqsubseteq S$ to a role box $\mathfrak{R}$ and consider the concept satisfiability w.r.t. $\mathfrak{R}$. Currently, neither $\mathcal{ALC}_{\mathcal{RA}^\ominus}$ nor $\mathcal{ALC}_{\mathcal{RA}}$ provide the identity role.

Other Fragments of FOPL: In the following we will briefly discuss whether decidability or undecidability of $\mathcal{ALC}_{\mathcal{RA}^\ominus}$ follows from already known results in logic, namely from results in bounded number of variables FOPL, or results from research carried out in the so-called (loosely) guarded fragment of FOPL. To the best of our knowledge, no previously known decidability resp. undecidability result is exploitable in the case of $\mathcal{ALC}_{\mathcal{RA}^\ominus}$.

It is well-known that certain fragments of FOPL are decidable, for example, the class of all closed FOPL formulas containing at most two variables, denoted by FO^2. FO^2 has the finite model property – each satisfiable formula has a finite model. We already noted that one would need at least three variables if one translates $\mathcal{ALC}_{\mathcal{RA}^\ominus}$ role boxes and concepts into FOPL. In fact, there is no way even to express the transitivity axiom $\forall x, y, z : R(x, y) \wedge R(y, z) \Rightarrow R(x, z)$ in FO^2 (see [10]). If FO^2 is augmented by transitivity on an extra-logical level (since transitivity cannot be expressed within the language itself), FO^2 becomes undecidable, as Grädel et al. have shown (see [10]). However, the class FO^2 is much too large to capture the concept-side of $\mathcal{ALC}_{\mathcal{RA}^\ominus}$, since $\mathcal{ALC}$ concepts are expressible in a proper subset of FO^2, namely GF^2_-, see below. Recall that $\mathcal{ALC}_{\mathcal{R}^+}$ *is* decidable.

[2] It follows that if $\mathcal{ALC}_{\mathcal{RA}^\ominus}$ was decidable it would be EXPTIME-hard, since $\mathcal{ALC}_\oplus$ is EXPTIME-complete.

The so-called *Guarded Fragment* (GF) as introduced by Andréka, van Benthem and Németi is another fragment of FOPL that is decidable (it even has the finite tree model property). We will not formally discuss it here (see [9, 8]). However, its prominent feature is that the number of variables is *not* bounded, as long as certain syntactic restrictions on the use of the quantifiers are obeyed. Grädel suggested to use the guarded fragment as the basis for a new family of n-ary DLs (see [8]). Since FO^3 is undecidable, but GF^3 (the guarded fragment with three variables) is decidable, decidability for $\mathcal{ALC}_{\mathcal{RA}^\ominus}$ would follow if $\mathcal{ALC}_{\mathcal{RA}^\ominus}$ was expressible in GF^3. A few informal words regarding the guarded fragment seem to be appropriate: when translating propositional modal logics (for example, $\mathcal{ALC}$ resp. K_n) into FOPL, one observes that the quantifiers are always used in a certain *guarded* way. The quantifiers appear only in "patterns" of the form $\forall x, y : R(x, y) \Rightarrow C(y)$ and $\exists x, y : R(x, y) \wedge C(y)$. Here, the atom $R(x, y)$ is used as a *guard.* This observation was generalized into the guarded fragment and observed to be responsible for the nice computational properties resp. decidability of many modal logics (and the guarded fragment as well). More specifically, the guard must always be an atom (complex formulas may not be guards) and must contain all variables that appear in the subsequent of the formula "behind" the guard. The formulas $\forall x, y : R(x, y) \Rightarrow C(y)$ and $\exists x, y : R(x, y) \wedge C(y)$ are therefore in GF^2, where GF^2 is the guarded fragment with two variables: $GF^2 = F0^2 \cap GF$. If all non-unary atoms are only used as guards, the GF formula is said to be *monadic.* This is obviously the case for $\mathcal{ALC}$, since the binary relations occur solely as guards. Obviously, the transitivity axiom $\forall x, y, z : R(x, y) \wedge R(y, z) \Rightarrow R(x, z)$ is *not* in the GF. The *loosely* guarded fragment (LGF) is a generalization of the guarded fragment by additionally allowing not only atoms (like $R(x, y)$) being guards, but also *conjunctions* of atoms. However, the transitivity axiom is not even in the LGF, since it is additionally required that there must be a conjunct using x and z in *one* guard atom; e.g. $\forall x, y, z : R(x, y) \wedge R(y, z) \wedge S(x, z) \Rightarrow R(x, z)$ is in the LGF, but $\forall x, y, z : R(x, y) \wedge R(y, z) \Rightarrow R(x, z)$ is not. Grädel has even shown that it is impossible to express that a relation is transitively closed within the guarded or the loosely guarded fragment. The transitivity axiom $\forall x, y, z : R(x, y) \wedge R(y, z) \Rightarrow R(x, z)$ cannot be expressed by any means (see [9]). But this means that $\mathcal{ALC}_{\mathcal{RA}^\ominus}$ is not in the LGF.

Therefore, the (loosely) guarded fragment has been extended by transitivity on an "extra-logical" level (since transitivity is not expressible within the logic itself), and the following results have been obtained:[3]

- GF^3 with transitive relations is undecidable (see [9]).
- LGF_- with one transitive relation is undecidable (see [7]).

[3] A minus suffix indicates that the logic does not provide equality.

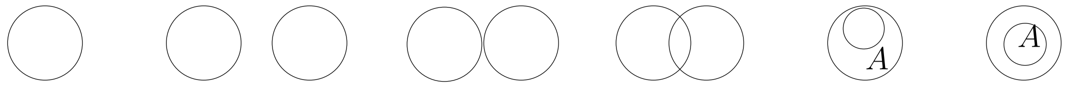

Figure 3: RCC8 Qualitative Spatial Relationships: EQ = Equal, DC = Disconnected, EC = Externally Connected, PO = Partial Overlap, TPP = Tangential Proper Part, $NTPP$ = Non-Tangential Proper Part. Read the relations as $TPP(A, B)$, $NTPP(A, B)$ etc. TPP and $NTPP$ have corresponding inverse relationships: $TPPI$ and $NTPPI$, e.g. $TPPI(B, A)$, $NTPPI(B, A)$.

- Even GF^2_- with transitive relations is undecidable (see [7]).
- *Monadic* GF^2_- with binary transitive, symmetric and/or reflexive relations is *decidable* (see [7]).

None of these results is applicable in the case of $\mathcal{ALC}_{\mathcal{RA}^\ominus}$. The most important result concerning $\mathcal{ALC}_{\mathcal{RA}^\ominus}$ is the last one, since $\mathcal{ALC}$ is in monadic GF^2_-, and the role box allows one to express, for example, transitivity. However, the role boxes of $\mathcal{ALC}_{\mathcal{RA}^\ominus}$ can express a lot more than transitivity. Therefore, this result implies the decidability of, e.g. $\mathcal{ALC}_{\mathcal{R}^+}$, but not of $\mathcal{ALC}_{\mathcal{RA}^\ominus}$. In fact, a much more general result has been shown by Ganzinger et al. (see [7]), but it does not apply to axioms of the form $\forall x, y, z : R(x, y) \wedge S(y, z) \Rightarrow T(x, z)$.

4 Spatial Reasoning With $\mathcal{ALC}_{\mathcal{RA}^\ominus}$

A widely accepted approach in the field of spatial reasoning for describing spatial relationships between two-dimensional objects in the plane is to describe their spatial interrelationship qualitatively instead of describing their metrical and/or geometrical attributes. Examples for qualitative spatial calculi fitting into this category are the well-known RCC8 calculus (see [16]) and the so-called Egenhofer-relations (see [6]). In the case of RCC8, we can distinguish 8 *disjoint* – pairwise exclusive – base relations that describe purely topological aspects of the scene, *exhaustively* covering the space of all possibilities (see Figure 3). Informally speaking this means that between every two objects in the plane *exactly one* of the RCC8 relations holds.

Given a set of base relations, e.g. the RCC8 relations, the most important inference problem is the following: given three regions a, b and c in the plane, and the relations $R(a, b), S(b, c)$ between them, what can be deduced about the possible relationships between a and c? This basic inference task is usually given by

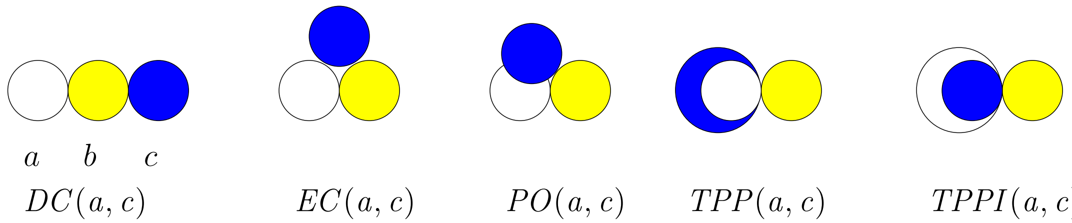

Figure 4: Illustration of $\forall a, b, c : EC(a, b) \wedge EC(b, c) \Rightarrow (DC(a, c) \vee EC(a, c) \vee PO(a, c) \vee TPP(a, c) \vee TPPI(a, c))$

means of a so called *composition table* that lists, given the "column" relationship $R(a, b)$ and the "row" relationship $S(b, c)$, all possible relationships $T_1(a, c)$, $T_2(a, c)$, ..., $T_n(a, c)$ that may hold between a and c. For example, in the case of RCC8, the composition table contains the entry $\{DC, EC, PO, TPP, TPPI\}$, given the relationship EC for the row as well as for the column – please consider Figure 4. This corresponds to the FOPL axiom $\forall a, b, c : EC(a, b) \wedge EC(b, c) \Rightarrow (DC(a, c) \vee EC(a, c) \vee PO(a, c) \vee TPP(a, c) \vee TPPI(a, c))$, which is equivalent to the role axiom $EC \circ EC \sqsubseteq DC \sqcup EC \sqcup PO \sqcup TPP \sqcup TPPI$.

Usually, also the *disjointness* of the base relations must be captured. As already noted, $\mathcal{ALC}_{\mathcal{RA}^\ominus}$ lacks this expressiveness (and it cannot be simulated by means of other constructs easily, see below), but $\mathcal{ALC}_{\mathcal{RA}}$ does not. For an adequate modeling of spatial relationships, also *inverse roles* must be taken into account. For example, the RCC8 relationship $TPPI$ is the inverse of TPP, and $NTPPI$ is the inverse of $NTPP$. Of course, $TPP^{\mathcal{I}} = (TPPI^{\mathcal{I}})^{-1}$ and $NTPP^{\mathcal{I}} = (NTPPI^{\mathcal{I}})^{-1}$ should be ensured. However, both $\mathcal{ALC}_{\mathcal{RA}}$ and $\mathcal{ALC}_{\mathcal{RA}^\ominus}$ lack inverse roles, since undecidability would follow immediately then by previously known undecidability results. Since we use $\mathcal{ALC}_{\mathcal{RA}^\ominus}$ in the following example, we can neither rely on $TPP^{\mathcal{I}} = (TPPI^{\mathcal{I}})^{-1}$ nor on the disjointness of roles.

The possibility to *approximate* composition tables, which are very widely used in the field of relation algebra-based knowledge representation and reasoning, is the distinguishing feature of $\mathcal{ALC}_{\mathcal{RA}^\ominus}$ and $\mathcal{ALC}_{\mathcal{RA}}$. Usually, the $\mathcal{ALC}_{\mathcal{RA}}$ approximation will be better, since the disjointness of the base relations is also enforced. Nevertheless, as the example demonstrates, we can still solve some interesting spatial reasoning task using $\mathcal{ALC}_{\mathcal{RA}^\ominus}$. Consider the following TBox:

$$
\begin{array}{lcl}
circle & \dot{\sqsubseteq} & figure \\
figure\_touching\_a\_figure & \doteq & figure \sqcap \exists EC.figure \\
special\_figure & \doteq & figure \sqcap \\
 & & \forall PO.\neg figure \sqcap \\
 & & \forall NTPPI.\neg figure \sqcap \\
 & & \forall TPPI.\neg circle \sqcap \\
 & & \exists TPPI.(figure \sqcap \exists EC.circle)
\end{array}
$$

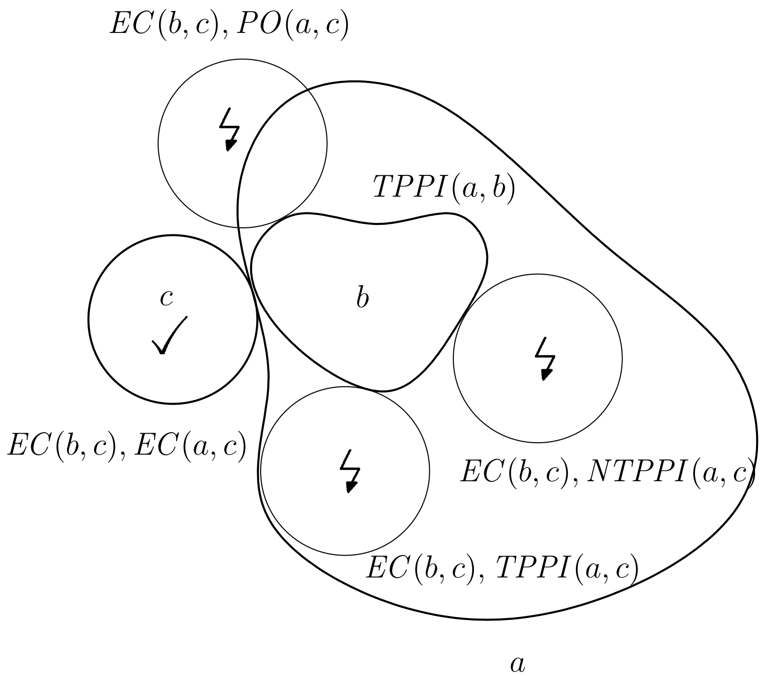

Figure 5: Illustration of a model of *special_figure*

Then, the question is: does $figure\_touching\_a\_figure$ subsume $special\_figure$, or equivalently, is $figure \sqcap \forall PO.\neg figure \sqcap \forall NTPPI.\neg figure \sqcap \forall TPPI.\neg circle \sqcap \exists TPPI.(figure \sqcap \exists EC.circle) \sqcap \neg(figure \sqcap \exists EC.figure)$ unsatisfiable w.r.t. a role box $\mathfrak{R}$ corresponding to the RCC8 composition table?

After pushing the negation sign inwards and removing the obviously contradictory disjunct from the resulting disjunction, the concept $figure \sqcap \forall PO.\neg figure \sqcap \forall NTPPI.\neg figure \sqcap \forall TPPI.\neg circle \sqcap \exists TPPI.(figure \sqcap \exists EC.circle) \sqcap \forall EC.\neg figure$ must be unsatisfiable then. Please consider Figure 5 which illustrates a "model" of $special\_figure$, with $a \in special\_figure^{\mathcal{I}}$, $b \in (figure \sqcap \exists EC.circle)^{\mathcal{I}}$, and $c \in circle^{\mathcal{I}}$, with $\langle a, b \rangle \in TPPI^{\mathcal{I}}$, $\langle b, c \rangle \in EC^{\mathcal{I}}$; please note that $TPPI \circ EC \sqsubseteq EC \sqcup PO \sqcup TPPI \sqcup NTPPI \in \mathfrak{R}$. Due to the definition of $special\_figure$, it can be seen that in every model $\langle a, c \rangle \in EC^{\mathcal{I}}$ must hold. But then, due to $\forall EC.\neg figure$, it is obviously the case that $figure \sqcap \forall PO.\neg figure \sqcap \forall NTPPI.\neg figure \sqcap \forall TPPI.\neg circle \sqcap \exists TPPI.(figure \sqcap \exists EC.circle) \sqcap \forall EC.\neg figure$ has no models and is therefore unsatisfiable. This shows that $special\_figure$ is indeed subsumed by $figure\_touching\_a\_figure$.

Please note that there are also other description logics suitable for spatial reasoning tasks, namely the language $\mathcal{ALCRP}(\mathcal{D})$ (see [11]). However, unrestricted $\mathcal{ALCRP}(\mathcal{D})$ is undecidable (see [15]), and its decidable fragment suffers from very strong syntax-restrictions, dramatically pruning the space of allowed concept expressions. In fact, the finite model property is ensured in restricted $\mathcal{ALCRP}(\mathcal{D})$. The strong syntactic requirements make modeling with $\mathcal{ALCRP}(\mathcal{D})$ much more complicated, and many interesting spatial-reasoning tasks cannot be addressed within the decidable fragment. One the other hand, the special $\mathcal{ALCRP}(\mathcal{D})$ instantiation $\mathcal{ALCRP}(\mathcal{S}_2)$ captures the semantics of the RCC8 spatial relationships much more appropriately than it would be possible with $\mathcal{ALC}_{\mathcal{RA}^\ominus}$– inverse roles are present and disjointness is ensured as well. For example, see [12] for an $\mathcal{ALCRP}(\mathcal{S}_2)$ spatial reasoning application. However, $\mathcal{ALCRP}(\mathcal{S}_2)$ suffers from the same strong syntax restrictions which nearly make it impossible to address more complex spatial reasoning tasks. One of the motivations for our work on $\mathcal{ALC}_{\mathcal{RA}}$ and $\mathcal{ALC}_{\mathcal{RA}^\ominus}$ was to create a logic that might be used more freely than $\mathcal{ALCRP}(\mathcal{S}_2)$ for spatial modeling and reasoning.

5 Proving Undecidability of $\mathcal{ALC}_{\mathcal{RA}^\ominus}$

The structure of the proof is as follows: first we show that the intersection problem for a special class of context-free grammars – so called concatenation grammars – is undecidable. Then we show that the intersection problem of concatenation grammars could be solved iff the satisfiability problem of $\mathcal{ALC}_{\mathcal{RA}^\ominus}$ was decidable. This obviously shows that the latter must be undecidable as well, since the former is. It should be noted that the underlying idea of the proof given below is nearly identical to the idea exploited in the proof given by Ganzinger et al. in [7] for showing the undecidability of LGF_{-} with one transitive relation. However, the proof has been found independently, and for different classes of languages ($\mathcal{ALC}_{\mathcal{RA}^\ominus}$ is not in LGF). We start with some basic definitions needed for the proofs:

Definition 9 (Context-Free Grammar, Language) A context-free grammar $\mathcal{G}$ is a quadruple $(\mathcal{V}, \Sigma, \mathcal{P}, S)$, where $\mathcal{V}$ is a finite set of variables or non-terminal symbols, Σ is finite alphabet of terminal symbols with $\mathcal{V} \cap \Sigma = \emptyset$, and $\mathcal{P} \subseteq \mathcal{V} \times (\mathcal{V} \cup \Sigma)^+$ is a set of productions or grammar rules. $S \in \mathcal{V}$ is the start variable. The *language* generated by a context-free grammar $\mathcal{G}$ is defined as $\mathcal{L}(\mathcal{G}) = \{\, w \mid w \in \Sigma^*, S \xrightarrow{*} w \,\}$ (see [21]). In the following, we will only consider languages with $\epsilon \notin \mathcal{L}(\mathcal{G})$ (therefore we write $\mathcal{L}(\mathcal{G}) = \{\, w \mid w \in \Sigma^+, S \xrightarrow{+} w \,\}$). □

Definition 10 (Intersection-Problem for Languages) Let $\mathcal{L}_1$ and $\mathcal{L}_2$ be formal languages (e.g. context-free languages). The *intersection problem* is to decide whether $\mathcal{L}_1 \cap \mathcal{L}_2$ is empty or not. □

For lack of a better name we will consider special context-free grammars that we call *concatenation grammars* (for reasons that will become clear later):

Definition 11 (Concatenation Grammar) A context-free grammar $\mathcal{G} = (\mathcal{V}, \Sigma, \mathcal{P}, S)$ is called a *concatenation grammar* iff $\mathcal{P} \subseteq \mathcal{V} \times ((\mathcal{V} \cup \Sigma) \times (\mathcal{V} \cup \Sigma))$. $\square$

We say that a language is a concatenation language iff it has a generating concatenation grammar. For example, the language $\{a, b\}$ is not a concatenation language. The language $\{\, a^n b^n \mid n \geq 1 \,\}$ is a concatenation language, since it is generated by the grammar $(\{S, X\}, \{a, b\}, \{S \to a\ b, S \to a\ X, X \to S\ b\}, S)$.

Lemma 1 The intersection problem for concatenation languages is undecidable. $\square$

Proof 1 (Thanks to Harald Ganzinger who has suggested this proof) Let $\mathcal{G}_1 = (\mathcal{V}_1, \Sigma_1, \mathcal{P}_1, S_1)$ and $\mathcal{G}_2 = (\mathcal{V}_2, \Sigma_2, \mathcal{P}_2, S_2)$ be two arbitrary context-free grammars in Chomsky Normal Form.[4] Let $\# \notin \mathcal{V}_1 \cup \mathcal{V}_2 \cup \Sigma_1 \cup \Sigma_2$ be a new terminal symbol, for $i \in \{1, 2\}$: $\Sigma'_i =_{def} \Sigma_i \cup \{\#\}$, $\mathcal{P'}_i =_{def} \{\, A \to B\ C \mid A \to B\ C \in \mathcal{P}_i \,\} \cup \{\, A \to a\# \mid A \to a \in \mathcal{P}_i \,\}$, and $\mathcal{G'}_i = (\mathcal{V}_i, \Sigma'_i, \mathcal{P'}_i, S_i)$.

Then, $\mathcal{G'}_1 \cap \mathcal{G'}_2 = \emptyset$ iff $\mathcal{G}_1 \cap \mathcal{G}_2 = \emptyset$. Since the latter is an undecidable problem for context-free grammars (e.g. see [21]), the former is undecidable as well. $\square$

Given an arbitrary concatenation grammar, the key-observation is now that one can simply *reverse* the productions $\mathcal{P}$ of the grammar and get a role box $\mathfrak{R}$. If a word can be derived "top down" by the grammar using a derivation tree, then it is possible to "parse" this word in a bottom-up style using the role axioms. The following Lemma fixes the relationship between words that are derivable by a concatenation grammar and the models of the role box corresponding to this grammar:

Lemma 2 Let $\mathcal{G} = (\mathcal{V}, \Sigma, \mathcal{P}, S)$ be an arbitrary concatenation grammar. Let $w = w_1 \ldots w_n$ be a word, $w \in \Sigma^+$ with $|w| \geq 2$, and $\mathcal{I}$ be a model of $(\exists w_1. \ldots \exists w_n. \top, \mathfrak{R})$ with $\mathfrak{R} =_{def} \{\, B \circ C \sqsubseteq A \mid A \to B\ C \in \mathcal{P} \,\}$. Let $\langle x_0, x_1 \rangle \in w_1^{\mathcal{I}}, \ldots \langle x_{n-1}, x_n \rangle \in w_n^{\mathcal{I}}$ be an arbitrary path in the model $\mathcal{I}$ corresponding to w.

Let $V \in \mathcal{V}$ be an arbitrary non-terminal of $\mathcal{G}$. Then, $\langle x_0, x_n \rangle \in V^{\mathcal{I}}$ holds in all models $\mathcal{I}$ of $(\exists w_1. \ldots \exists w_n. \top, \mathfrak{R})$ iff there is a derivation of w having V as the root node: we write $V \xrightarrow{+} w$. As a consequence, $\langle x_0, x_n \rangle \in S^{\mathcal{I}}$ in all models $\mathcal{I}$ of $(\exists w_1. \ldots \exists w_n. \top, \mathfrak{R})$ iff $w \in \mathcal{L}(\mathcal{G})$. $\square$

Proof 2 **"$\Leftarrow$"** This can be shown using induction over the length of w.

[4] A context-free grammar $\mathcal{G} = (\mathcal{V}, \Sigma, \mathcal{P}, S)$ is in Chomsky Normal Form, iff $\mathcal{P} \subseteq \mathcal{V} \times ((\mathcal{V} \times \mathcal{V}) \cup \Sigma))$ (see [21]).

- If $|w| = 2$, $w = w_1 w_2$, and $V \xrightarrow{+} w$, then there must be a production of the form $V \to w_1 w_2 \in \mathcal{P}$. Note that there cannot be productions of the form $V \to w_1 B$, $V \to A w_2$, $V \to AB$, since $\mathcal{G}$ is a concatenation grammar – we would additionally need productions of the form $A \to w_1 \ldots$, or even productions of the form $A \to \epsilon$. If $\mathcal{I}$ is a model of $\mathfrak{R}$ and $<x_0, x_1> \in w_1^{\mathcal{I}}$, $<x_1, x_2> \in w_2^{\mathcal{I}}$, then, due to $w_1 \circ w_2 \sqsubseteq V \in \mathfrak{R}$ we have $<x_0, x_2> \in V^{\mathcal{I}}$ in every model $\mathcal{I}$.
- Let $w = w_1 \ldots w_n$, $n \geq 3$. Let $V \xrightarrow{+} w$. Since $\mathcal{G}$ is a concatenation grammar, there must be a production of the form $V \to XY \in \mathcal{P}$, and the following cases can occur:
  1. $X \in \mathcal{V}$, $Y \in \Sigma$: then, there is a derivation $X \xrightarrow{+} w_1 \ldots w_{n-1}$, and $Y = w_n$. Due to the induction hypothesis we have $<x_0, x_{n-1}> \in X^{\mathcal{I}}$ in every model $\mathcal{I}$. Since we consider a model of $(\exists w_1 . \ldots \exists w_{n-1} . \exists w_n . \top, \mathfrak{R})$, with $<x_{n-1}, x_n> \in w_n^{\mathcal{I}}$, we have $<x_0, x_n> \in V^{\mathcal{I}}$, because $\mathcal{I}$ is a model of $\mathfrak{R}$ with $X \circ w_n \sqsubseteq V \in \mathfrak{R}$.
  2. $X \in \Sigma$, $Y \in \mathcal{V}$: same argumentation.
  3. $X \in \mathcal{V}$, $Y \in \mathcal{V}$: let $w = uv$ be the partition of w corresponding to the derivations $X \xrightarrow{+} u$, $Y \xrightarrow{+} v$. Let $u = w_1 \ldots w_i$, $v = w_{i+1} \ldots w_n$. Due to the induction hypothesis we have $<x_0, x_i> \in X^{\mathcal{I}}$ and $<x_{i+1}, x_n> \in Y^{\mathcal{I}}$, since both u and v have a length smaller than n. We have $X \circ Y \sqsubseteq V \in \mathfrak{R}$. This shows that $<x_0, x_n> \in V^{\mathcal{I}}$.

  Summing up we have shown that $<x_0, x_n> \in V^{\mathcal{I}}$ holds in all models $\mathcal{I}$ of $(\exists w_1 . \ldots \exists w_n . \top, \mathfrak{R})$, if $V \xrightarrow{+} w$.

"⇒" If $<x_0, x_n> \in V^{\mathcal{I}}$ holds in all models $\mathcal{I}$ of $(\exists w_1 . \ldots \exists w_n . \top, \mathfrak{R})$, then the presence of $<x_0, x_n> \in V^{\mathcal{I}}$ is a *logical consequence* of $(\exists w_1 . \ldots \exists w_n . \top, \mathfrak{R})$. Therefore, $<x_0, x_n> \in V^{\mathcal{I}}$ is enforced by the role axioms in $\mathfrak{R}$. One can easily construct a derivation tree for w, showing that $V \xrightarrow{+} w$, by inspecting one of these models. More formally this could be shown using induction as well, and the proof would be very similar to the previous one.

□

Since we are trying to reduce the intersection problem of concatenation grammars to the satisfiability problem of $\mathcal{ALC}_{\mathcal{RA}^\ominus}$, we have to deal with *two* grammars. Please note that concatenation grammars are *not* closed under intersection (i.e. for two grammars $\mathcal{G}_1$ and $\mathcal{G}_2$ there is in general no concatenation grammar $\mathcal{G}_{1,2}$ such that $\mathcal{L}(\mathcal{G}_{1,2}) = \mathcal{L}(\mathcal{G}_1) \cap \mathcal{L}(\mathcal{G}_2)$). In order to deal with this problem we have two put *two* concatenation grammars into one role box:

Lemma 3 Let $\mathcal{G}_1 = (\mathcal{V}_1, \Sigma_1, \mathcal{P}_1, S_1)$ and $\mathcal{G}_2 = (\mathcal{V}_2, \Sigma_2, \mathcal{P}_2, S_2)$ be two arbitrary concatenation grammars. Without loss of generality can assume $\mathcal{V}_1 \cap \mathcal{V}_2 = \emptyset$, since we can always consistently rename the variables in one of the grammars and get $\mathcal{V}_1 \cap \mathcal{V}_2 = \emptyset$.

For $i \in \{1, 2\}$, we define $\mathfrak{R}_{\mathrm{i}} =_{def} \{ B \circ C \sqsubseteq A \mid A \to B\ C \in \mathcal{P}_i \}$.
Let $\Sigma =_{def} \Sigma_1 \cup \Sigma_2$ and $\mathfrak{R} =_{def} \mathfrak{R}_1 \cup \mathfrak{R}_2$.

Then, for $i \in \{1, 2\}$, $w \in \mathcal{L}(\mathcal{G}_i)$ iff $<x_0, x_n> \in S_i^{\mathcal{I}}$ in all models $\mathcal{I}$ $\mathcal{I}$ of $(\exists w_1. \ldots \exists w_n.\top, \mathfrak{R})$. Obviously, $w \in \mathcal{L}(\mathcal{G}_1) \cap \mathcal{L}(\mathcal{G}_2)$ iff $<x_0, x_n> \in S_1^{\mathcal{I}} \cap S_2^{\mathcal{I}}$ in all models $\mathcal{I}$ of $(\exists w_1. \ldots \exists w_n.\top, \mathfrak{R})$. □

Proof 3 An easy consequence of the previous Lemma and of the requirement that $\mathcal{V}_1 \cap \mathcal{V}_2 = \emptyset$ (the derivation trees do not become "mixed", i.e. each grammar solely uses its own productions). □

As an application of this Lemma, let us consider the two grammars

- $\mathcal{G}_1 = (\{S_1\}, \{a, b\}, \mathcal{P}_1, S_1)$, where
  $\mathcal{P}_1 = \{S_1 \to ab \mid aS_1b\}$,
- $\mathcal{G}_2 = (\{S_2\}, \{a, b\}, \mathcal{P}_2, S_2)$, where
  $\mathcal{P}_2 = \{S_2 \to aabb \mid aaS_2bb\}$.

Obviously, $\mathcal{L}(\mathcal{G}_1) = \{ a^n b^n \mid n \geq 1 \}$ and $\mathcal{L}(\mathcal{G}_2) = \{ a^{2n} b^{2n} \mid n \geq 1 \}$. Transformed into concatenation grammars we get

- $\mathcal{G'}_1 = (\{S_1, A\}, \{a, b\}, \mathcal{P'}_1, S_1)$, where
  $\mathcal{P'}_1 = \{S_1 \to ab \mid aA, A \to S_1b\}$, and
- $\mathcal{G'}_2 = (\{S_2, B, C, D, E, F\}, \{a, b\}, \mathcal{P'}_2, S_2)$, where
  $\mathcal{P'}_2 = \{S_2 \to aB, B \to aC, C \to bb,$
  $S_2 \to aD, D \to aE, E \to S_2F, F \to bb\}$.

The corresponding role box is

$$\begin{aligned}\mathfrak{R} = \ & \{\ a \circ b \sqsubseteq S_1, a \circ A \sqsubseteq S_1, S_1 \circ b \sqsubseteq A\ \} \ \cup \\ & \{\ a \circ B \sqsubseteq S_2, a \circ C \sqsubseteq B, b \circ b \sqsubseteq C, \\ & \ \ a \circ D \sqsubseteq S_2, a \circ E \sqsubseteq D, S_2 \circ F \sqsubseteq E, b \circ b \sqsubseteq F\ \}.\end{aligned}$$

The "first part" of this role box corresponds to $\mathcal{P'}_1$, and the "second part" to $\mathcal{P'}_2$. The symbols of the grammars correspond to roles now. Please consider $(\forall S_1.C \sqcap \forall S_2.D \sqcap \exists a.\exists a.\exists b.\exists b.\neg(C \sqcap D), \mathfrak{R})$. Any model of $(\forall S_1.C \sqcap \forall S_2.D \sqcap \exists a.\exists a.\exists b.\exists b.\neg(C \sqcap D), \mathfrak{R})$ would also be a model of $(\exists a.\exists a.\exists b.\exists b.\top, \mathfrak{R})$, and must therefore contain $<x_0, x_4> \in S_1^{\mathcal{I}} \cap S_2^{\mathcal{I}}$, because $w = aabb \in \mathcal{L}(\mathcal{G'}_1) \cap \mathcal{L}(\mathcal{G'}_2)$, due to Lemma 3. Since $x_0 \in (\forall S_1.C \sqcap \forall S_2.D)^{\mathcal{I}}$ also

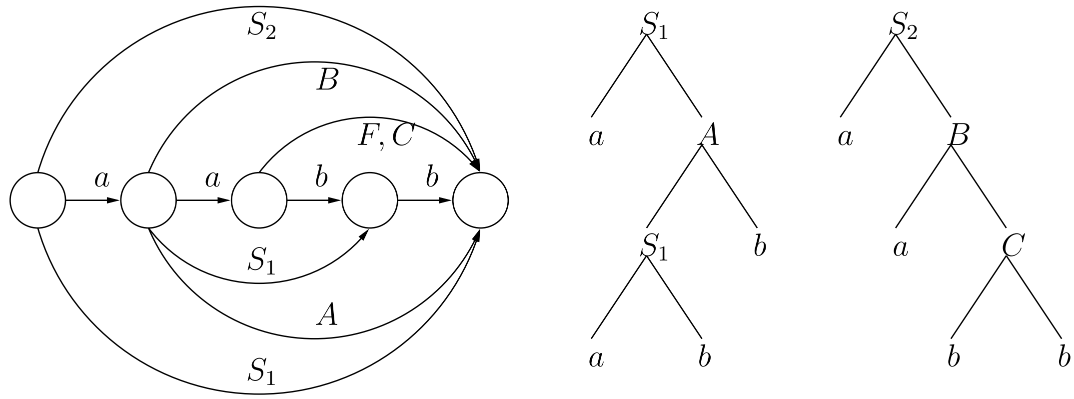

Figure 6: “Bottom up parsing” of $aabb \in \mathcal{L}(\mathcal{G}_1) \cap \mathcal{L}(\mathcal{G}_2)$

$x_4 \in (C \sqcap D)^{\mathcal{I}}$ must hold, which obviously contradicts $x_4 \in (\neg(C \sqcap D))^{\mathcal{I}}$. The example is therefore unsatisfiable. Considering Figure 6, it can be seen that the role box performs a “bottom up parsing” of the word $aabb$ – the two derivation trees shown in the figure can be immediately discovered as role compositions in the graph.

We can now prove the main result of this section by showing how to reduce the intersection problem of concatenation grammars to the satisfiability problem of $\mathcal{ALC}_{\mathcal{RA}^\ominus}$:

Theorem 1 The satisfiability problem of $\mathcal{ALC}_{\mathcal{RA}^\ominus}$ is undecidable. $\square$

Proof 4 We give an example for a pair $(E, \mathfrak{R})$ for which no algorithm exists that is capable of checking its satisfiability.

Let $\mathcal{G}_1 = (\mathcal{V}_1, \Sigma_1, \mathcal{P}_1, S_1)$ and $\mathcal{G}_2 = (\mathcal{V}_2, \Sigma_2, \mathcal{P}_2, S_2)$ be two arbitrary concatenation grammars. Without loss of generality we assume $\mathcal{V}_1 \cap \mathcal{V}_2 = \emptyset$.

For $i \in \{1, 2\}$, we define $\mathfrak{R}_i =_{def} \{\, B \circ C \sqsubseteq A \mid\; A \rightarrow B\ C \in \mathcal{P}_i \,\}$.
Let $\Sigma =_{def} \Sigma_1 \cup \Sigma_2$ and $\mathfrak{R} =_{def} \mathfrak{R}_1 \cup \mathfrak{R}_2$. Let $R_? \notin \mathsf{roles}(\mathfrak{R})$, and let

$$\mathfrak{R}' =_{def} \mathfrak{R} \cup \{\, R \circ S \sqsubseteq R_? \mid\; R, S \in (\{R_?\} \cup \mathsf{roles}(\mathfrak{R})),\ \neg\exists ra \in \mathfrak{R} : \mathsf{pre}(ra) = (R, S) \,\}$$

be the *completion* of $\mathfrak{R}$.

Then, $(E, \mathfrak{R}')$ is satisfiable iff $\mathcal{L}(\mathcal{G}_1) \cap \mathcal{L}(\mathcal{G}_2) = \emptyset$, where

$$E =_{def} X \sqcap \neg(C \sqcap D) \sqcap Y \sqcap \forall S_1.C \sqcap \forall S_2.D, \text{ with}$$

$$X =_{def} \sqcap_{a \in \Sigma} \exists a.\top \text{ and}$$

$$Y =_{def} \sqcap_{R \in \mathsf{roles}(\mathfrak{R}')} \forall R.(X \sqcap \neg(C \sqcap D)).$$

Since $\mathcal{L}(\mathcal{G}_1) \cap \mathcal{L}(\mathcal{G}_2) = \emptyset$ is undecidable, the satisfiability of $(E, \mathfrak{R}')$ is undecidable as well.

We have to show that $(E, \mathfrak{R}')$ is satisfiable iff $\mathcal{L}(\mathcal{G}_1) \cap \mathcal{L}(\mathcal{G}_2) = \emptyset$:

$\Rightarrow$ We prove the contra-positive: if $\mathcal{L}(\mathcal{G}_1) \cap \mathcal{L}(\mathcal{G}_2) \neq \emptyset$, then $(E, \mathfrak{R}')$ is unsatisfiable. Assume to the contrary that $\mathcal{L}(\mathcal{G}_1) \cap \mathcal{L}(\mathcal{G}_2) \neq \emptyset$, but $(E, \mathfrak{R}')$ is satisfiable. Let $\mathcal{I}$ be a model of $(E, \mathfrak{R}')$. Because $\mathcal{I}$ satisfies $\mathfrak{R}'$, it holds that $<x_0, x_n> \in (\bigcup_{R \in \mathsf{roles}(\mathfrak{R}')} R^{\mathcal{I}})^+$ implies $<x_0, x_n> \in *^{\mathcal{I}}$, where $*^{\mathcal{I}} =_{def} \bigcup_{R \in \mathsf{roles}(\mathfrak{R}')} R^{\mathcal{I}}$ is the so-called *universal relation.* This is ensured by the fact that the composition of two arbitrary roles from $\mathsf{roles}(\mathfrak{R}')$ is always defined in $\mathfrak{R}'$, due to the completion process. Since $\mathcal{I}$ is a model of E, there is some $x_0 \in E^{\mathcal{I}}$. Due to $x_0 \in (X \sqcap Y)^{\mathcal{I}}$ it holds that $x_0 \in ((\sqcap_{a \in \Sigma} \exists a.\top) \sqcap (\sqcap_{R \in \mathsf{roles}(\mathfrak{R}')} \forall R.(\sqcap_{a \in \Sigma} \exists a.\top)))^{\mathcal{I}}$. The model $\mathcal{I}$ therefore represents *all* possible words $w \in \Sigma^+$. Let $w \in \mathcal{L}(\mathcal{G}_1) \cap \mathcal{L}(\mathcal{G}_2)$, with $w = w_1 \ldots w_{n-1} w_n$. Obviously, $\mathcal{I}$ is also a model of $\exists w_1. \ldots \exists w_n.\top$, with $x_0 \in (\exists w_1. \ldots \exists w_n.\top)^{\mathcal{I}}$. Let $<x_0, x_1> \in w_1^{\mathcal{I}}$, $\ldots <x_{n-1}, x_n> \in w_n^{\mathcal{I}}$ be a path in the model corresponding to w; $<x_0, x_n> \in *^{\mathcal{I}}$ holds. If $\mathcal{I}$ if a model of $\mathfrak{R}'$, then it is also a model of $\mathfrak{R}$ due to $\mathfrak{R} \subseteq \mathfrak{R}'$, and therefore Lemma 3 is applicable. Due to Lemma 3 we then have $<x_0, x_n> \in S_1^{\mathcal{I}} \cap S_2^{\mathcal{I}}$ in *every* model, and since $x_0 \in (\forall S_1.C \sqcap \forall S_2.D)^{\mathcal{I}}$, $x_n \in (C \sqcap D)^{\mathcal{I}}$ must also hold in every model. However, this obviously contradicts $x_n \in \neg(C \sqcap D)^{\mathcal{I}}$ which must hold because $<x_0, x_n> \in *^{\mathcal{I}}$ and $x_0 \in (\sqcap_{R \in \mathsf{roles}(\mathfrak{R}')} \forall R.\neg(C \sqcap D))^{\mathcal{I}}$. This shows that there are no models. $(E, \mathfrak{R}')$ is therefore unsatisfiable.

$\Leftarrow$ If $\mathcal{L}(\mathcal{G}_1) \cap \mathcal{L}(\mathcal{G}_2) = \emptyset$, then we show that $(E, \mathfrak{R}')$ is satisfiable by constructing an infinite model. The model $\mathcal{I}$ is constructed incrementally, e.g. $\mathcal{I}_0 \subset \mathcal{I}_1 \subset \mathcal{I}_2 \subset \cdots \subset \mathcal{I}_\omega$, $\mathcal{I} = \mathcal{I}_\omega$. We refer to the set $\bigcup_{a \in \Sigma} a^{\mathcal{I}}$ as the *skeleton* of the model $\mathcal{I}$. The skeleton has the form of an infinite tree. An illustration of $\mathcal{I}$ is given in Figure 7; the thick lines correspond to the skeleton. Each node in the model $\mathcal{I}$ has $|\Sigma|$ different *direct successors* in the skeleton; the skeleton of $\mathcal{I}$ is a tree with branching factor $|\Sigma|$.

For each $n \in \mathbb{N} \cup \{0\}$, the skeleton of the interpretation $\mathcal{I}_n$ is a tree of depth n, encoding *all* words w with $|w| \leq n$, i.e. $w \in \bigcup_{i \in \{0,\ldots,n\}} \Sigma^i$. Each word w of length $i = |w|$, $i \leq n$, corresponds to a path from the root node $x_{0,0}$ to some node $x_{i,m}$ at depth i, in all skeletons of the models $\mathcal{I}_n$. Therefore, the skeleton of $\mathcal{I}$ represents *all* words from Σ^+.

Intuitively, the terminal symbols of the words to be parsed by the role box are represented as *direct* edges in the skeleton of the model, whereas the *indirect* edges in this model are inserted to mimic the "bottom-up parsing process" of these words, which is performed by the role box. The construction of $\mathcal{I}$ therefore works as follows:

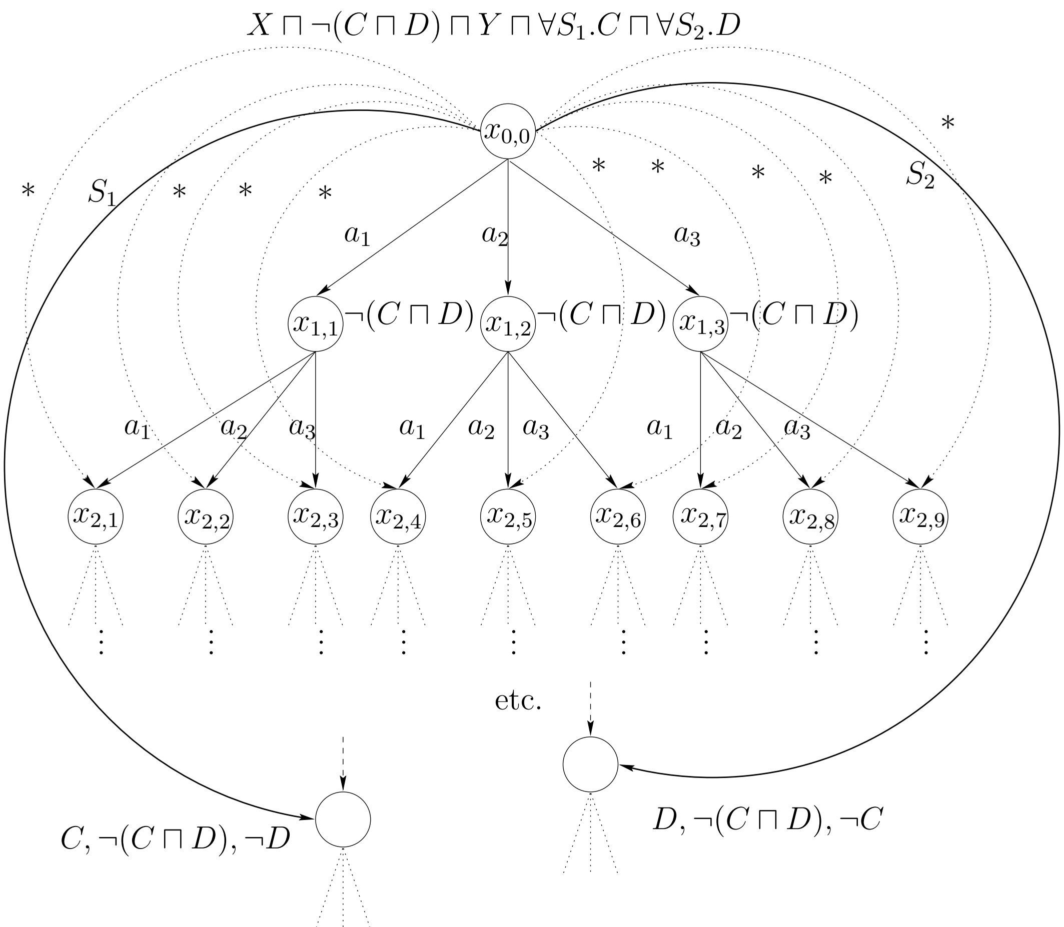

Figure 7: Illustration of the constructed model for $(E, \mathfrak{R}')$

- $\mathcal{I}_0 = (\Delta^{\mathcal{I}}_0, \cdot^{\mathcal{I}}_0)$, $\Delta^{\mathcal{I}}_0 := \{x_{0,0}\}$, $\cdot^{\mathcal{I}}_0 := \{\}$
- For $n \in 0, 1, \ldots$,
  $\mathcal{I}_{n+1} = (\Delta^{\mathcal{I}}_{n+1}, \cdot^{\mathcal{I}}_{n+1})$ is constructed from $\mathcal{I}_n = (\Delta^{\mathcal{I}}_n, \cdot^{\mathcal{I}}_n)$ as follows:
  1. $\Delta^{\mathcal{I}}_{n+1} := \Delta^{\mathcal{I}}_n \cup \{\, x_{n+1,j} \mid j \in \{1, \ldots, |\Sigma|^{n+1}\} \,\}$,
     $\cdot^{\mathcal{I}}_{n+1} := \cdot^{\mathcal{I}}_n$
  2. $\Sigma = \{a_1, \ldots, a_k\}$, $\forall a_r \in \{a_1, \ldots, a_k\}$:
     $$a_r^{\mathcal{I}_{n+1}} := a_r^{\mathcal{I}_{n+1}} \cup \{\, <x_{n,j}, x_{n+1,k(j-1)+r}> \mid x_{n,j} \in \Delta^{\mathcal{I}}_n, x_{n+1,k(j-1)+r} \in \Delta^{\mathcal{I}}_{n+1} \,\}$$
  3. `while` $\mathcal{I}_{n+1} \not\models \mathfrak{R}'$ `do`
     `for each` $R \circ S \sqsubseteq T \in \mathfrak{R}'$ `do`
     $T^{\mathcal{I}_{n+1}} := T^{\mathcal{I}_{n+1}} \cup R^{\mathcal{I}_{n+1}} \circ S^{\mathcal{I}_{n+1}}$
     `od`
     `od`

4. $C^{\mathcal{I}_{n+1}} := C^{\mathcal{I}_{n+1}} \cup \{ x_{n+1,j} \mid <x_{0,0}, x_{n+1,j}> \in S_1^{\mathcal{I}_{n+1}} \}$
5. $D^{\mathcal{I}_{n+1}} := D^{\mathcal{I}_{n+1}} \cup \{ x_{n+1,j} \mid <x_{0,0}, x_{n+1,j}> \in S_2^{\mathcal{I}_{n+1}} \}$

$\mathcal{I}$ is a model: due to step 3 in the construction we have $\mathcal{I}_n \models \mathfrak{R}'$ for all $n \in \mathbb{N} \cup \{0\}$ (since we have a finite number of role axioms and $\mathcal{I}_n$ is finite as well, the `while`-loop terminates in finite time), and therefore obviously $\mathcal{I} \models \mathfrak{R}'$. We prove that $x_{0,0} \in E^{\mathcal{I}}$. Due to the construction it is obviously the case that $x_0 \in ((\sqcap_{a \in \Sigma} \exists a.\top) \sqcap (\sqcap_{R \in \mathsf{roles}(\mathfrak{R}')} \forall R.(\sqcap_{a \in \Sigma} \exists a.\top)) \sqcap \forall S_1.C \sqcap \forall S_2.D)^{\mathcal{I}}$: for each node $x_{i,j} \in \Delta^{\mathcal{I}}$, we have $<x_{0,0}, x_{i,j}> \in *^{\mathcal{I}}$ (recall that $*^{\mathcal{I}} =_{def} \bigcup_{R \in \mathsf{roles}(\mathfrak{R}')} R^{\mathcal{I}}$), and each node has the required $k = |\Sigma|$ successors, $a_1, \ldots, a_k$. This holds for $x_{0,0}$ as well as for $x_{i,j}$. This shows that $x_{0,0} \in X^{\mathcal{I}}$, $x_{i,j} \in X^{\mathcal{I}}$, and therefore $x_{0,0} \in (\forall R.(\sqcap_{a \in \Sigma} \exists a.\top))^{\mathcal{I}}$. It also holds that $x_{0,0} \in (\sqcap_{R \in \mathsf{roles}(\mathfrak{R}')} \forall R.\neg(C \sqcap D))^{\mathcal{I}}$. Assume the contrary: then there must be some successor node $x_{n,i_n} \in \Delta^{\mathcal{I}}$ with $x_{n,i_n} \in C^{\mathcal{I}}$, $x_{n,i_n} \in D^{\mathcal{I}}$. Since this node lies at depth n, it holds that $x_{n,i_n} \in \Delta^{\mathcal{I}_n}$ with $x_{n,i_n} \in C^{\mathcal{I}_n}$, $x_{n,i_n} \in D^{\mathcal{I}_n}$. Due to the construction, $x_{n,i_n} \in C^{\mathcal{I}_n}$ iff $<x_{0,0}, x_{n,i_n}> \in S_1^{\mathcal{I}_n}$, and $x_{n,i_n} \in D^{\mathcal{I}_n}$ iff $<x_{0,0}, x_{n,i_n}> \in S_2^{\mathcal{I}_n}$. Let w be the corresponding path of length n in the skeleton with $w = w_1 \ldots w_n$, $<x_{0,0}, x_{1,i_1}> \in w_1^{\mathcal{I}}, \ldots <x_{n-1,i_{n-1}}, x_{n,i_n}> \in w_n^{\mathcal{I}}$, with $w_i \in \{a_1, a_2, i_1, \ldots, i_k\}$, leading from $x_{0,0}$ to x_{n,i_n}. But then, due to Lemma 3, $w \in \mathcal{L}(\mathcal{G}_1) \cap \mathcal{L}(\mathcal{G}_2)$, due to $<x_{0,0}, x_{n,i_n}> \in S_1^{\mathcal{I}_n} \cap S_2^{\mathcal{I}_n}$. Contradiction. Summing up we have shown that $\mathcal{I} \models (E, \mathfrak{R}')$. □

6 Discussion & Conclusion

We have proven that the satisfiability problem of $\mathcal{ALC}_{\mathcal{RA}^\ominus}$ is undecidable. As already noted, this is a severe result, due to the high relevance of axioms having the form $S \circ T \sqsubseteq R_1 \sqcup \cdots \sqcup R_n$ in the field of qualitative (spatial or temporal) reasoning.

Considering the proof, it can be seen that not the whole expressiveness of $\mathcal{ALC}$ was needed in order to show the undecidability. In fact, the existential restrictions used in the proof have only the form $\exists R.\top$, no *qualified* existential restrictions were needed ($\exists R \equiv \exists R.\top$). Additionally, we did not make use of disjunctions on the right hand side of the role axioms in the proof – all role axioms were of the form $R \circ S \sqsubseteq T$. The negation operator was only used within E in the form $\neg(C \sqcap D)$, which can be rewritten as $\neg C \sqcup \neg D$. Therefore, no full negation operator is needed; it is sufficient if the DL provides negation for concept names. Summing up, the language $\mathcal{ALU}$ with *deterministic* role boxes, called $\mathcal{ALU}_{\mathcal{RA}^\ominus}$, is already undecidable.[5]

[5] $\mathcal{ALU}$ provides negation for concept names, disjunction $\sqcup$, universal qualification $\forall R.C$

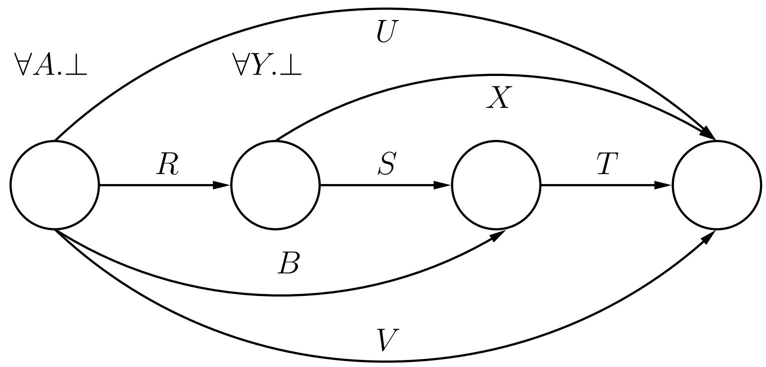

Figure 8: Illustration of an $\mathcal{ALC}_{\mathcal{RA}^\ominus}$ model of $\mathfrak{R}$ and $\exists R.((\exists S.\exists T.\top) \sqcap \forall Y.\bot) \sqcap \forall A.\bot$. The same concept is unsatisfiable in $\mathcal{ALC}_{\mathcal{RA}}$ w.r.t. $\mathfrak{R}$.

It is obvious that special kinds of role boxes lead to decidability. For example, if we restrict the set of admissible role boxes to role boxes of the form $\{R_1 \circ R_1 \sqsubseteq R_1, \ldots, R_n \circ R_n \sqsubseteq R_n\}$, we get a syntactic variation of the logic $\mathcal{ALC}_{\mathcal{R}^+}$, with $R_1 \ldots R_n$ declared as transitively closed roles. However, before considering special role boxes that might yield decidability and therefore special $\mathcal{ALC}_{\mathcal{RA}^\ominus}$ fragments, it is very important to understand the principal limitations which was the motivation for this investigation. The question remains whether admissible role boxes can be found which are still useful for spatial reasoning tasks. Obviously, the syntactic restrictions should not be stronger than necessary. For example, considering a syntactic variation of $\mathcal{ALC}_{\mathcal{R}^+}$ makes no sense.

In the following we will only briefly sketch why the undecidabiliy result given here does not immediately apply to $\mathcal{ALC}_{\mathcal{RA}}$. Let us examine the difference between $\mathcal{ALC}_{\mathcal{RA}}$ and $\mathcal{ALC}_{\mathcal{RA}^\ominus}$. Considering the two different but very similar looking languages, the question arises whether $\mathcal{ALC}_{\mathcal{RA}}$ is in fact subsumed by $\mathcal{ALC}_{\mathcal{RA}^\ominus}$.

The language $\mathcal{ALC}_{\mathcal{RA}}$ requires that all roles have to be interpreted as disjoint: for any two different roles R, S, $R \neq S$ and any interpretation $\mathcal{I}$, $R^\mathcal{I} \cap S^\mathcal{I} = \emptyset$ must hold. The calculus given in [22] requires that the role boxes are *unique*: for any pair of roles R, S, there is at most one role axiom $ra \in \mathfrak{R}$ with $\mathsf{pre}(ra) = (R, S)$. The disjointness for roles really makes a difference: for example, the concept $\exists R.((\exists S.\exists T.\top) \sqcap \forall Y.\bot) \sqcap \forall A.\bot$ w.r.t. $\mathfrak{R} = \{R \circ S \sqsubseteq A \sqcup B, S \circ T \sqsubseteq X \sqcup Y, A \circ T \sqsubseteq U, B \circ T \sqsubseteq V, R \circ X \sqsubseteq U, R \circ Y \sqsubseteq V\}$ is satisfiable in $\mathcal{ALC}_{\mathcal{RA}^\ominus}$, but unsatisfiable in $\mathcal{ALC}_{\mathcal{RA}}$ (see Figure 8). This is due to the fact that a non-empty role intersection between U and V is enforced, violating the disjointness requirement, yielding unsatisfiability only in the case of $\mathcal{ALC}_{\mathcal{RA}}$.

$\mathcal{ALC}_{\mathcal{RA}}$ cannot be easily reduced to $\mathcal{ALC}_{\mathcal{RA}^\ominus}$, even though $\mathcal{ALC}_{\mathcal{RA}^\ominus}$ seems to

and *unqualified* existential quantification $\exists R$.

be the more general description logic, since there are less restrictions. If $(C, \mathfrak{R})$ is satisfiable in $\mathcal{ALC}_{\mathcal{RA}}$, then it is also satisfiable in $\mathcal{ALC}_{\mathcal{RA}^\ominus}$ as well, but not vice versa. One idea to enforce role disjointness within $\mathcal{ALC}_{\mathcal{RA}^\ominus}$ might be the following: for each role pair R, S with $R \neq S$ to be declared as disjoint, create a new atomic "marker concept", e.g. $\boxed{RS}$, and add two universal value restrictions like $\forall R.\boxed{\mathrm{RS}} \sqcap \forall S.\neg\boxed{\mathrm{RS}}$ conjunctively to the original concept C. $(C, \mathfrak{R})$ would be transformed into $(C \sqcap \forall R.\boxed{\mathrm{RS}} \sqcap \forall S.\neg\boxed{\mathrm{RS}} \sqcap \cdots, \mathfrak{R})$, for each pair of disjoint roles R, S. Let $\mathcal{I}$ be a model of the latter, and let $x_0 \in (C \sqcap \forall R.\boxed{\mathrm{RS}} \sqcap \forall S.\neg\boxed{\mathrm{RS}} \sqcap \cdots)^{\mathcal{I}}$. Unfortunately, this only ensures that $(\{ \langle x_0, x_i \rangle \mid x_i \in \Delta^{\mathcal{I}} \} \cap R^{\mathcal{I}} \cap S^{\mathcal{I}}) = \emptyset$, which is obviously a much too weak requirement. As a "solution" one might think that the universal role * might be used in order to propagate $\forall R.\boxed{\mathrm{RS}} \sqcap \forall S.\neg\boxed{\mathrm{RS}} \sqcap \cdots$ to *every* individual in the model. $(C, \mathfrak{R})$ would be transformed into $(C \sqcap \forall R.\boxed{\mathrm{RS}} \sqcap \forall S.\neg\boxed{\mathrm{RS}} \sqcap \cdots \sqcap \forall * .(\forall R.\boxed{\mathrm{RS}} \sqcap \forall S.\neg\boxed{\mathrm{RS}} \sqcap \cdots), \mathfrak{R}')$. Unfortunately, this is a much stronger requirement than disjointness for roles, since the additional conjunct now enforces $\{ x_j \mid \langle x_i, x_j \rangle \in R^{\mathcal{I}} \} \cap \{ x_j \mid \langle x_i, x_j \rangle \in S^{\mathcal{I}} \} = \emptyset$, which obviously implies $R^{\mathcal{I}} \cap S^{\mathcal{I}} = \emptyset$.

Instead, one would need some kind of "counting construct" that would enable the distinction of different individuals in order to simulate the role disjointness of $\mathcal{ALC}_{\mathcal{RA}}$ within $\mathcal{ALC}_{\mathcal{RA}^\ominus}$. We therefore believe that disjoint roles are really something very special that cannot be simulated by means of any $\mathcal{ALC}_{\mathcal{RA}^\ominus}$ construction. Therefore, we conjecture that $\mathcal{ALC}_{\mathcal{RA}}$ is not subsumed by $\mathcal{ALC}_{\mathcal{RA}^\ominus}$.

In the undecidability proof we enforced the existence of the appropriate successors $a_1, \ldots, a_k$ with $\Sigma = \{a_1, \ldots, a_k\}$ for every node in the model. The existence of every word w in the model was therefore granted. For this purpose the role box was completed, using the auxiliary role $R_?$. In the case of $\mathcal{ALC}_{\mathcal{RA}}$, we can make use of the same construction. However, due to the disjointness requirement, things get more complicated.

Considering the reduction, the unsatisfiability is due to the fact that $\forall S_1.C \sqcap \forall S_2.D$ is used to assert C and D to *one and the same* individual from the root node in order to produce an inconsistency via $\neg(C \sqcap D)$ which holds for all individuals. Obviously, with a disjointness requirement on S_1 and S_2, the presence of S_1 and S_2 connecting the nodes $x_{0,0}$ and $x_{i,j}$ such that $\langle x_{0,0}, x_{i,j} \rangle \in S_1^{\mathcal{I}} \cap S_2^{\mathcal{I}}$ is an inconsistency by itself. Therefore, $\forall S_1.C \sqcap \forall S_2.D$ is useless in order to yield an inconsistency, since the inconsistency is present in the first place due to the disjointness requirement. If the same proof technique could be applied for showing the undecidability of $\mathcal{ALC}_{\mathcal{RA}}$, then neither disjunctions nor atomic negation would be needed for this undecidability proof: if $X =_{def} \exists a_1 \sqcap \cdots \sqcap \exists a_k$, then $(X \sqcap (\sqcap_{R \in \mathsf{roles}(\mathfrak{R}')} \forall R.X), \mathfrak{R}')$ would suffice to prove the undecidability! This concept is already expressible in the language $\mathcal{FL}^-$, showing even the undecidability of $\mathcal{FL}^-_{\mathcal{RA}}$. This would indicate that composition-based role axioms are really *highly* problematic language constructs, since adding them to one of the simplest of all description logics $\mathcal{FL}^-$ would already yield undecidability. However, it is an open question whether the exploited proof technique can indeed be applied.

Considering the tableaux calculus for $\mathcal{ALC}_{\mathcal{RA}}$ given in [22], we only conjectured that $\mathcal{ALC}_{\mathcal{RA}}$ might be decidable. We did not prove it. The given tableaux calculus was incomplete since it suffered from the definition of a so-called *blocking condition*. The tableaux calculus was presented in the expectation that an appropriate blocking condition could be found in the future. However, we still have not found a correct blocking condition for $\mathcal{ALC}_{\mathcal{RA}}$. On the other hand, the reader should be informed that we have also carefully tried to reduce various other known undecidable problems to $\mathcal{ALC}_{\mathcal{RA}}$, but without success (e.g. the Domino Problem).

As the investigation has shown, the exact position of the boundary line between decidable and undecidable description logics with composition-based role axioms of the form $S \circ T \sqsubseteq R_1 \sqcup \cdots \sqcup R_n$ must be investigated much more thoroughly in the future.

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References

[1] F. Baader. Augmenting concept languages by transitive closure of roles: An alternative to terminological cycles. In *Twelfth International Conference on Artificial Intelligence, Darling Harbour, Sydney, Australia, Aug. 24-30, 1991*, pages 446–451, August 1991.

[2] R.J. Brachman and J.G. Schmolze. An overview of the KL-ONE knowledge representation system. *Cognitive Science*, pages 171–216, August 1985.

[3] M. Buchheit, F.M. Donini, and A. Schaerf. Decidable reasoning in terminological knowledge representation systems. *Journal of Artificial Intelligence Research*, 1:109–138, 1993.

[4] F.M. Donini, M. Lenzerini, D. Nardi, and W. Nutt. The complexity of concept languages. Technical Report RR-95-07, German Center for AI (DFKI), 1995.

[5] F.M. Donini, M. Lenzerini, D. Nardi, and A. Schaerf. Reasoning in description logics. In G. Brewka, editor, *Principles of Knowledge Representation.* CSLI Publications, 1996.

[6] M.J. Egenhofer. Reasoning about binary topological relations. In O. Günther and H.-J. Schek, editors, *Advances in Spatial Databases, Second Symposium, SSD'91, Zurich, Aug. 28-30, 1991*, volume 525 of *Lecture Notes in Computer Science*, pages 143–160. Springer Verlag, Berlin, August 1991.

[7] H. Ganzinger, Chr. Meyer, and M. Veanes. The two-variable guarded fragment with transitive relations. In *Proc. 14th IEEE Symposium on Logic in Computer Science*, pages 24–34. IEEE Computer Society Press, 1999. To appear in LICS'99.

[8] E. Grädel. Guarded fragments of first-order logic: a perspective for new description logics? Extended abstract, Proceedings of 1998 International Workshop on Description Logics DL '98, Trento 1998, CEUR Electronic Workshop Proceedings, `http://sunsite.informatik.rwth-aachen.de/Publications/CEUR-WS/Vol-11`.

[9] E. Grädel. On the restraining power of guards. (24 pages), to appear in Journal of Symbolic Logic.

[10] E. Grädel, M. Otto, and E. Rosen. Undecidability Results for Two-Variable Logics. *Archive for Mathematical Logic*, 38:313–354, 1999. See also: Proceedings of 14th Symposium on Theoretical Aspects of Computer Science STACS'97, Lecture Notes in Computer Science No. 1200, Springer 1997, 249–260.

[11] V. Haarslev, C. Lutz, and R. Möller. A description logic with concrete domains and a role-forming predicate operator. *Journal of Logic and Computation*, 9(3):351–384, June 1999.

[12] V. Haarslev, R. Möller, A.-Y. Turhan, and M. Wessel. On terminological default reasoning about spatial information: Extended abstract. In P. Lambrix et al., editor, *Proceedings of the International Workshop on Description Logics (DL'99), July 30 - August 1, 1999, Linköping, Sweden*, pages 155–159, June 1999.

[13] I. Horrocks and U. Sattler. A description logic with transitive and inverse roles and role hierarchies. *Journal of Logic and Computation*, 9(3):385–410, 1999.

[14] I. Horrocks, U. Sattler, and S. Tobies. Practical reasoning for expressive description logics. In *Proceedings of the 6th International Conference on Logic for Programming and Automated Reasoning (LPAR'99)*, 1999.

[15] C. Lutz and R. Möller. Defined topological relations in description logics. In M.-C. Rousset et al., editor, *Proceedings of the International Workshop on Description Logics, DL'97, Sep. 27-29, 1997, Gif sur Yvette, France*, pages 15–19. Universite Paris-Sud, Paris, September 1997.

[16] D. A. Randell, Z. Cui, and A. G. Cohn. A Spatial Logic based on Regions and Connections. In B. Nebel, C. Rich, and W. Swartout, editors, *Principles of Knowledge Representation and Reasoning*, pages 165–176, 1992.

[17] U. Sattler. A concept language extended with different kinds of transitive roles. In G. Görz and S. Hölldobler, editors, *20. Deutsche Jahrestagung für Künstliche Intelligenz*, number 1137 in Lecture Notes in Artificial Intelligence, pages 333–345. Springer Verlag, Berlin, 1996.

[18] K. Schild. A correspondence theory for terminological logics: Preliminary report. In *Twelfth International Conference on Artificial Intelligence, Darling Harbour, Sydney, Australia, Aug. 24-30, 1991*, pages 466–471, August 1991.

[19] M. Schmidt-Schauß. Subsumption in KL-ONE is Undecidable. In *Principle of Knowledge Representation and Reasoning – Proceedings of the First International Conference KR '89*, 1989.

[20] M. Schmidt-Schauß and G. Smolka. Attributive concept descriptions with complements. *Artifical Intelligence*, 48:1–26, 1991.

[21] U. Schöning. *Theoretische Informatik kurz gefasst*. BI-Wissenschaftsverlag, 1. edition, 1992.

[22] M. Wessel, V. Haarslev, and R. Möller. $\mathcal{ALC}_{\mathcal{RA}}$ – $\mathcal{ALC}$ with Role Axioms. In F. Baader and U. Sattler, editors, *Proceedings of the International Workshop in Description Logics 2000 (DL2000)*, number 33 in CEUR-WS, pages 21–30, Aachen, Germany, August 2000. RWTH Aachen. Proceedings online available from `http://SunSITE.Informatik.RWTH-Aachen.DE/Publications/CEUR-WS/Vol-33/`.

[23] W.A. Woods and J.G. Schmolze. The KL-ONE family. In F. Lehmann, editor, *Semantic Networks in Artificial Intelligence*, pages 133–177. Pergamon Press, Oxford, 1992.