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Qualitative Spatial Reasoning with the $\mathcal{ALCI}_{\mathcal{RCC}}$ Family - First Results and Unanswered Questions[1]

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Qualitative Spatial Reasoning with the $\mathcal{ALCI}_{\mathcal{RCC}}$ Family - First Results and Unanswered Questions[†]

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1 Introduction and Motivation

In this report we introduce a family of description logics (DLs) suitable for qualitative spatial representation and reasoning tasks. We demonstrate the usefulness of the proposed DLs for representing and reasoning about qualitative spatial phenomena with an example from the realm of deductive Geographic Information Systems (GIS). The main contribution of this report is the presentation of some preliminary complexity results regarding the complexity of the satisfiability problem in the considered extended DLs. We show three of the considered logics to be decidable; for the others we give coarse lower complexity bounds. We also outline the relationships to modal logics and give a sound and complete axiomatization of the proposed DLs in hybrid modal logics.

The report is aimed to give a "state-of-the-art" description of what *we* know about the considered issues, in terms of a progress report. Please read further to see what the considered issues are. We think that the obtained new results should be preserved in this report to provide a basis for future research on these topics, even though the results are still somewhat preliminary yet.

The report is structured as follows: first we give a short informal introduction into the basics of description logic. Readers familiar with DLs can safely skip this introduction. Then we informally introduce the main ideas of DL-based qualitative spatial reasoning and pinpoint some requirements that a DL must fulfill in order to be useful for qualitative spatial reasoning tasks. The core assumptions of the approach are presented. We outline the relationships to our previous work, as well as the theoretical contribution of this paper. We then define the considered DLs $\mathcal{ALCI}_{\mathcal{RCC}1}$, $\mathcal{ALCI}_{\mathcal{RCC}2}$, $\mathcal{ALCI}_{\mathcal{RCC}3}$, $\mathcal{ALCI}_{\mathcal{RCC}5}$ and $\mathcal{ALCI}_{\mathcal{RCC}8}$, their syntax and semantics, and state some obvious consequences from these definitions. In the next chapter we illustrate the usefulness of $\mathcal{ALCI}_{\mathcal{RCC}5}$ in a deductive geographic information system scenario by means of reasoning with an ontology containing spatio-thematic geographic concepts.

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The main contribution of this report are the preliminary theoretical results presented in the then following section (see also [27]). In terms of a state-of-the-art report we fix what we know and what we don't know yet about the logics in the $\mathcal{ALCI}_{\mathcal{RCC}}$ family. We show decidability of three of the considered DLs ($\mathcal{ALCI}_{\mathcal{RCC}1}$, $\mathcal{ALCI}_{\mathcal{RCC}2}$, and $\mathcal{ALCI}_{\mathcal{RCC}3}$), and give coarse lower complexity bounds for the remaining logics ($\mathcal{ALCI}_{\mathcal{RCC}5}$ and $\mathcal{ALCI}_{\mathcal{RCC}8}$), whose decidability status is unknown. We then fix the relationships to modal logics and give a sound and complete axiomatization of the $\mathcal{ALCI}_{\mathcal{RCC}}$ family in hybrid modal logics. We make some remarks on finite model reasoning and discuss some problems and ideas related to this. Finally, future work is outlined.

1.1 Description Logics in General - a Short Introduction

Description logics (DLs), also called concept languages, are nowadays an accepted standard for decidable knowledge representation frameworks (see [1]). As the term *description* logics indicates, these representation languages offer the ability to *describe* (resp. *model*) concepts from an application domain, in a *logic-based* representation language. A *concept* might be anything we have a notion of. In order to *represent* such a concept in a DL, this notion must be somehow definable in a formal, *logic-based* way. DLs offer a simple and variable-free concept description language with a simple syntax. Since DLs are based on formal logic, not only their syntax, but also their semantics as well as the offered *inference services* are well defined. However, their most prominent feature is their *decidability.* This means that the core inference problems in these logics (for example, the satisfiability problem) are decidable. Even though there are many undecidable knowledge representation frameworks that are actually used in applications (e.g., languages like PROLOG, implemented variants of full first-order predicate logic, etc.), the common understanding in the DL community is that practical knowledge representation systems should be based on decidable logics.

However, the developer of a DL-based system should also try to fulfill the needs in expressivity that users demand, if possible without loosing decidability. The trade-off between expressivity ("What can be said?") vs. decidability in every formal system is well-known. A broad family of DLs offering a wide range of expressive modeling constructs exists, and many sound, complete and terminating calculi for solving the basic decision problems in these languages are known ([4, 3]). Highly-optimized *implementations* of these calculi are known, turning description *logics* into efficient DL-based knowledge representation *systems* in the average case. Systems like RACER and FACT) can form the KR&R backbone for demanding applications that require a substantial amount of inferencing power (i.e., need to perform logical deduction) in order to solve the application problems.

Generally, a DL *concept language* offers concepts and roles. *Concepts* are defined inductively with the help of other concepts and roles. Sometimes, also roles can be defined from other roles, or somehow set into relation with other roles. As a rule of thumb, concepts correspond to *first-order formula with one free variable* - in its simplest form (in the case of an atomic concept, also called concept name), a concept is just a unary predicate symbol with a free variable, whereas *roles* correspond to first-order formulas with two free variables - in its simplest form, a role is just a binary predicate symbol (also called a *role name* or *role symbol*). Let us illustrate this with an example: the concept "father" might be defined/described as a "male human who has a child." "Male, human" and "father" correspond to unary predicates, whereas the relation "has child" gives rise to a binary predicate symbol. We can use the standard DL $\mathcal{ALC}$ (see [25]) to model this concept:[1]

$$father =_{def} human \sqcap male \sqcap (\exists has\_child.\top) \sqcap (\forall has\_child.human).$$

Note that *father* is just a syntactic name for the concept expression $human \sqcap male \sqcap (\exists has\_child.\top) \sqcap (\forall has\_child.human)$. The concept "$\top$" is the so-called "Top concept" representing the entire universe of discourse, $\Delta^{\mathcal{I}}$. Note that $\exists has\_child.\top \wedge \forall has\_child.human$ actually implies $\exists has\_child.human$.

The translation of *father* into a first-order predicate logic (FOPL) formula would yield

$$\begin{aligned} father[x] =_{def} male(x) \wedge human(x) \wedge\ & \exists y : has\_child(x, y) \wedge \top(y) \wedge \\ & \forall y : has\_child(x, y) \rightarrow human(y). \end{aligned}$$

Please note that x is the free variable, and that the existential and universal quantifiers are always used in a certain *guarded* way; i.e., binary predicate symbols (role names) are used as guards.

A "father having only sons" could now be defined as

$$father\_having\_only\_sons =_{def} father \sqcap \forall has\_child.male.$$

Note that $father$ must be syntactically replaced by the concept given above. The translation into FOPL of *father_having_only_sons* would yield

$$father\_having\_only\_sons[x] =_{def} father(x) \wedge \forall y : has\_child(x, y) \rightarrow male(y).$$

Sometimes, also *inverse roles* are needed:

$$happy\_child =_{def} human \sqcap \forall has\_parent.(rich \sqcap nice\_person).$$

We know that the role *has_parent* should be the *converse (inverse)* of the role *has_child*, and vice versa. To enforce the corresponding relationship between the interpretations of *has_parent* and *has_child*, we could use a *closed* FOPL statement of the form

$$\forall x, y : has\_child(x, y) \leftrightarrow has\_parent(y, x).$$

Extending $\mathcal{ALC}$ with inverse roles would yield the logic $\mathcal{ALCI}$ (where $\mathcal{I}$ indicates the presence of *inverse roles*).

Despite the fact that the translation of a DL into FOPL is not the most natural translation available (DL concepts are more closely related to modal logic formulas, see below) we have chosen to do so since FOPL can somehow be seen as a "lingua franca" and therefore offers a commonly accessible starting point.

The central inference problem in any logic is the *satisfiability problem*; in case of a DL,

[1] The standard DL $\mathcal{ALC}$ offers full concept negation, conjunction, disjunction, as well a as certain restricted kind of "guarded" existential and universal quantification over roles. For the given example, $\mathcal{ALC}$ is more than sufficient - even smaller DLs, e.g. $\mathcal{ALE}$, would be sufficient to model the example concepts given here. Concepts in $\mathcal{ALC}$ are equivalent to formulas in a certain modal logic which is called *multi-modal* $K_{(n)}$.

the *concept satisfiability problem* - given a concept C, is there an *interpretation* $\mathcal{I}$ that makes C true, i.e. can there be an individual i such that $C(x)$ holds if we interpret x as i? Such an interpretation is also called a *model*, and we say that C is *satisfied* by $\mathcal{I}$, written $\mathcal{I} \models C$. The individuals i can be seen as a witness of the satisfiability of C: $C^{\mathcal{I}} \neq \emptyset$ due to $i \in C^{\mathcal{I}}$. If there can be no such i, then C is called unsatisfiable. Given a model $\mathcal{I}$ of C, the set of individuals $C^{\mathcal{I}} \subseteq \Delta^{\mathcal{I}}$ is called the *extension* of C.

DLs also offer so-called *terminologies* or *TBoxes.* TBoxes are used to model the vocabulary of the application domain. Such a *conceptualization of an application domain* is also called an *ontology.* In its most general form, a TBox is simply a set of universally quantified closed implication statements of the form $\forall x : C_1[x] \rightarrow D_1[x]$, where C_1 and D_1 are FOPL translations of concepts (having a free variable x). In DL syntax, a terminology is written as a set of inclusion statements (also called TBox *axioms*) of the form $\{C_1 \dot{\sqsubseteq} D_1, \ldots, C_n \dot{\sqsubseteq} D_n\}$. Informally, a TBox $\{C_1 \dot{\sqsubseteq} D_1, \ldots, C_n \dot{\sqsubseteq} D_n\}$ enforces that the extensions of the C_i's must be subsets of the extensions of D_i's, in any model $\mathcal{I}$, such that for each i we have $C_i^{\mathcal{I}} \subseteq D_i^{\mathcal{I}}$. In this case, $\mathcal{I}$ is called a model of the TBox - it *satisfies* the TBox and therefore all its axioms. The TBox axiom $C \dot{\equiv} D$ is just a shorthand for $\{C \dot{\sqsubseteq} D, D \dot{\sqsubseteq} C\}$. Don't confuse $C \dot{\equiv} D$ with $C =_{def} D$: the former enforces a *semantic constraint on the models*, namely $C^{\mathcal{I}} = D^{\mathcal{I}}$ for all models of the TBox, whereas the latter has to be understood as a meta-logical macro definition on a *syntactical level* with the meaning of "whenever you encounter C, replace it with D". However, certain classes of TBoxes can indeed be understood as sets of macro definitions, see below. Concept satisfiability *w.r.t. a terminology* then means satisfiability of this concept by a model that also satisfies the TBox (such that all TBox axioms are satisfied).

Returning to our example, we could use a TBox to state that a "man" is a "male human", and vice versa. This gives us the TBox

$$\{man \dot{\sqsubseteq} human \sqcap male, \quad human \sqcap male \dot{\sqsubseteq} man.\}$$

Note that, without this TBox, the concept $man \sqcap \neg human$ would be *satisfiable*, since the extensions of *man* and *human* would not be set into relation somehow. But the TBox enforces that $man^{\mathcal{I}} = (human^{\mathcal{I}} \cap male^{\mathcal{I}})$, and therefore $(man \sqcap \neg human)^{\mathcal{I}}$ whose interpretation is $man^{\mathcal{I}} \cap (\Delta^{\mathcal{I}} \setminus human^{\mathcal{I}})$ is empty for every interpretation $\mathcal{I}$, demonstrating the required unsatisfiability (note that $\Delta^{\mathcal{I}}$ is the entire interpretation domain).

A TBox is called *simple* if the following restrictions are satisfied: the C_i's must be concepts names, each concept name must appear at most once on the left hand sides of the axioms, and there are no *cyclic definitions* in the TBox. In this case, the concept expressions on the right hand sides of the TBox axioms can then be seen as the *definitions* of the concept names on their left hand sides. An axiom of the form $C_i \dot{\sqsubseteq} D_i$ is called a *primitive concept definition* (giving *necessary conditions* for concept membership), whereas $C_i \dot{\equiv} D_i$ is called a *concept definition* (providing *sufficient and necessary conditions* for concept membership). Basically, primitive concept definitions like $C_i \dot{\sqsubseteq} D_i$ are taken as abbreviations for true concept definitions $C_i \dot{\equiv} D_i \sqcap D_i^{\star}$, where $D_i^{\star}$ is a freshly created atomic concept that doesn't appear elsewhere in the TBox, representing the *primitive part of the concept.* A simple TBox is also called an *unfoldable TBox,* since each defined concept name can be expanded by recursively replacing its defined concepts with their definitions until only concept names remain in the expression. Basically, this can be understood as a *macro expansion process*, very similar to the ones found in programming languages like LISP. For simple TBoxes, the semantical ("$C \doteq D$") and the syntactical notions ("$C =_{def} D$") coincide, but this is not the case for general TBoxes: if the C_i and D_i in the TBox axioms can be arbitrary concepts, then the DL admits so-called *general TBoxes*, and the TBox axioms are called *GCIs – general concept inclusion axioms.* General TBoxes are the most expressive ones, but have the drawback that they can (in general) no longer be understood as sets of concept definitions.

Concept names that are defined (used) in the TBox can be inserted automatically into a *subsumption hierarchy,* called the *taxonomy,* which is a lattice-like directed acyclic graph (DAG). In order to do so, the *subsumption problem* must be solved: decide whether a concept D is more general than a concept C. In this case, for all interpretations $\mathcal{I}$, $C^{\mathcal{I}} \subseteq D^{\mathcal{I}}$ holds. Considering the example it is clear that $father \sqsubseteq man$ and $father \sqsubseteq human$ should hold. Since $father =_{def} man \sqcap \ldots$ we see that $father \sqsubseteq human$ holds either due to the presence of $man \dot{\sqsubseteq} human$ in the TBox, or due to the (syntactic) definition of *man.* However, most of the computed subsumption relationships will not be explicitly modeled in such a way. It is one of the advantages a DL-based system has to offer that the taxonomy is actually computed by the system, and not modeled by the user. The concepts *man*, *human* are called *subsumers* of *father*, and *father* is called a *subsumee* of *man* and *human*. *father* would be called a *direct subsumee* of *man*; and *man* a *direct subsumer* of *father*. The corresponding closed FOPL formulas, e.g. $\forall x : father(x) \rightarrow man(x)$, can also be seen as logical consequences of the *theory of the knowledge base.* A DL-based system computes the most specific (direct) subsumers and subsumees for each concept name from a TBox. The Top-element in this (not necessarily complete!) "lattice" is denoted by $\top$ ($\top^{\mathcal{I}} = \Delta^{\mathcal{I}}$), and the Bottom-element by $\bot$ ($\bot^{\mathcal{I}} = \{\}$).

Moreover, in order to represent knowledge concerning a certain *specific state of the world,* the so-called *assertional Box, or ABox*, offers the ability to represent knowledge concerning individual objects and their interrelationships. For example, to say that "John" is a father, and that "Jim" is his child, we would use two *ABox axioms* - a so-called *concept membership assertion* of the form $john : father$, as well as a *role membership assertion* of the form $(john, jim) : has\_child$. The resulting ABox would be

$$\{john : father, \quad (john, jim) : has\_child\}.$$

In its simplest form, an ABox is simply a set of concept and role membership assertions. Depending on the DL, there might be other types of ABox axioms available. From a FOPL perspective, *john* and *jim* would be *constant symbols,* and the example ABox would be equivalent to the closed FOPL formula $father(john) \wedge has\_child(john, jim)$.

There are various inference problems centered around the ABox:

- The *ABox satisfiability problem* (possibly w.r.t. a TBox): given an ABox (and probably also a TBox), check whether this ABox is satisfiable, e.g. is $\{john :$

$father \sqcap \neg\exists has\_child.\top\}$ consistent?

- The *instance checking problem*: check whether a certain individual in an ABox is an instance of a specific concept (probably w.r.t. a TBox); e.g. can "Jim" be shown to be an instance of *human* if we have the ABox $\{john : father, \ (john, jim) : has\_child\}$ and the TBox $\{father \dot{\sqsubseteq} man \sqcap (\forall has\_child.human) \ldots\}$?

- The *instance retrieval problem*: given an ABox and a concept (possibly defined in a TBox), return the set of all ABox individuals that can be shown to be instances of that concept; e.g. return the set of all $father$s that are mentioned in the ABox.

- The *instance realization problem*: given an ABox individual and a TBox, compute the individual's most specific concept names from the TBox of which it is an instance of; e.g. for the ABox $\{john : father \wedge \forall has\_child.(male \sqcap human)\}$ this would be $father\_having\_only\_sons$ for $john$.

In the rest of the paper we will not use ABoxes extensively. But they are surely important components of a DL-based system and therefore deserved some attention here. Actually, there are various other services and inference problems that a DL-based knowledge representation system might offer. In fact, we only sketched the most important *standard inference services*. However, we want to close our short informal introduction on DLs here and ask the reader to refer to some of the much more elaborate available introductions e.g. the introductory chapters in the "Description Logic Handbook" ([1]). We barely scratched the surface here. However, we wanted to provide some basic background knowledge on DLs here in order to make the report more self-contained and accessible.

1.2 Description Logics for Qualitative Spatial Reasoning?

The previous examples illustrated the usage of DLs with the help of common-sense-like concepts like "father", "son", etc. But the long-term goal of this research is to create a description logic which supports the modeling of *qualitative spatial concepts*. In order to model them adequately with a DL, it turns out that *special expressiveness* is needed. For the previous examples we used the standard DL $\mathcal{ALCI}$. Unfortunately, the expressiveness of $\mathcal{ALCI}$ is insufficient for qualitative spatial reasoning tasks.

A *qualitative spatial concept* may be any spatial notion (of a spatial object, scene, configuration, ...) we have which can sufficiently be described using a *qualitative vocabulary*. A qualitative spatial description abstracts away from most of the present spatial aspects of the considered object (scene, configuration, ...) and focuses on certain *selected qualitative aspects* which are considered as being relevant for the given representation and reasoning task. For example, instead of specifying the exact size of a certain spatial object in terms of measured physical units, it might be sufficient to say that the object is just "big" or "small". From our point of view, quantitative vs. qualitative is mainly a matter of the *granularity or coarseness* of the available description vocabulary.

Generally, a *qualitative description* makes only as many distinctions as necessary to solve a specific application problem. This might not only reduce the amount of storage needed for the representation, but also the time complexity of the reasoning tasks. Moreover, qualitative spatial descriptions can be communicated more easily to humans since they are closer to human language. However, formalizing these - often vague - notions can be very hard. Different calculi are available for reasoning about different spatial aspects (form, orientation, size, configuration, ...). A whole research community develops qualitative spatial description languages and reasoning calculi for these. The field is called *Qualitative Spatial Reasoning (QSR).*

In this work, we focus on qualitative spatial concepts making use of *binary qualitative spatial relationships of relative locations.* Informally, we may assume that the considered qualitative spatial relationships are available as *roles* in a DL now. For example, we might want to say that two objects are overlapping, disjoint from each other, that one is contained within the other, and the like. We then want to define concepts like the concept of a "German lake" as being *the set of all lakes which are contained within the country Germany.* Here we have the notion of spatial containment. We might also want to define the concept of a "city at a lake", the notion of a "neighbor city", etc. Notions of adjacency, overlap, separateness etc. might crop up. In oder to capture the *inherent properties* of these qualitative spatial relationships (describing relative locations) as roles in a DL, we need very special expressiveness in a DL.

To give a more concrete example, suppose we want to define a *global integrity constraint* in a geographic information system that states that *countries never contain other countries.*[2] This could be achieved with a TBox statement of the form

$$country \dot{\sqsubseteq} \forall contains.\neg country.$$

Suppose that the ABox now contains

$$\{germany : country, x123 : country,$$
$$(germany, hamburg) : contains, (hamburg, x123) : contains\}.$$

Obviously, the modeled world is inconsistent, since *hamburg* contains a country $x123$. An inherent property of the containment relationship is its *transitivity* which means that $(germany, x123) : contains$ should hold. Even though this is not explicitly modeled in the ABox, it should be a logical consequence:

$$\{(germany, hamburg) : contains, (hamburg, x123) : contains\} \models$$
$$\{(germany, x123) : contains\}$$

The inconsistency due to the violation of $country \dot{\sqsubseteq} \forall contains.\neg country$ should surely be detected. This simple example shows that inherent properties of qualitative spatial relationships must be adequately captured. For the containment relationship, at least transitivity must be taken for granted. Other requirements might crop up as well – for example, the containment relationship should also be irreflexive and asymmetric - *contains* should be interpreted as a *strict partial order (SPO).*

[2] We are aware of certain exceptions from this "rule of thumb".

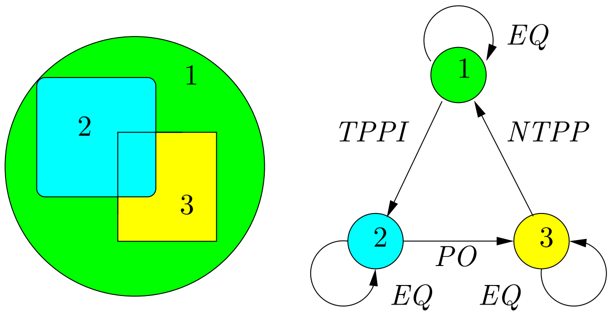


Figure 1: A spatial scene (left) and its corresponding complete edge-colored graph, describing the scene qualitatively (right). Edges are labeled with RCC8 relationships.

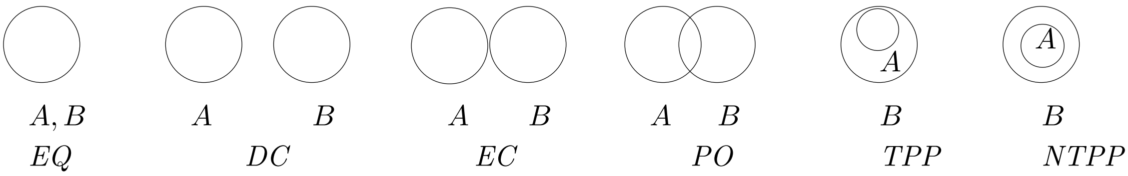


Figure 2: RCC8 Relationships, f.l.t.r.: Equal, Disconnected, Externally Connected, Partial Overlap, Tangential Proper Part, Non-Tangential Proper Part. To be read as $EQ(A, B)$, etc. All but *TPP* and *NTPP* are self-inverse. The inverses of *TPP* and *NTPP* are *TPPI* and *NTPPI*.

Suppose we want to describe the pairwise relative positions of objects in a concrete two-dimensional configuration in a qualitative way. Figure 1 gives an illustration of a a concrete scene of three two-dimensional objects, as well as a qualitative spatial description of the pairwise relative object positions in this scene. Each node in this complete graph (a complete graph with n nodes is also called a K_n) represents a scene object, and between any two nodes, *exactly one* so-called base-relation holds. This is called the *JEPD-property of the base relationships*: they are *jointly exhaustive and pairwise disjoint*. Obviously, a pair of objects can never be overlapping as well as adjacent, etc. This pairwise disjointness of relationships gives rise to the requirement that *disjoint roles* should be available as a modeling construct in a DL supporting qualitative spatial reasoning.

Some well-known and popular *sets of qualitative spatial relationships* are given by the so-called *RCC-family* of spatial reasoning calculi (see [22] for the origins, and [9] for an extensive survey). In the case of *RCC8,* we can distinguish eight base-relationships, see Figure 2. A coarser version called RCC5 is derived from RCC8 by collapsing the *TPPI* and *NTPPI* (*TPP* and *NTPP*) relationships into the RCC5 relationship *PP*, which is just an other name for the relationship *spatially inside* (resp. *PPI*, also simply called *spatially contains*), as well as joining *EC* and *DC* into *DR*. It even makes sense to define *RCC3, RCC2,* and *RCC1,* offering coarser and coarser qualitative spatial description vocabulary. As already noted, the granularity resp. descriptive power of these frameworks is determined by the coarseness of their basic description language

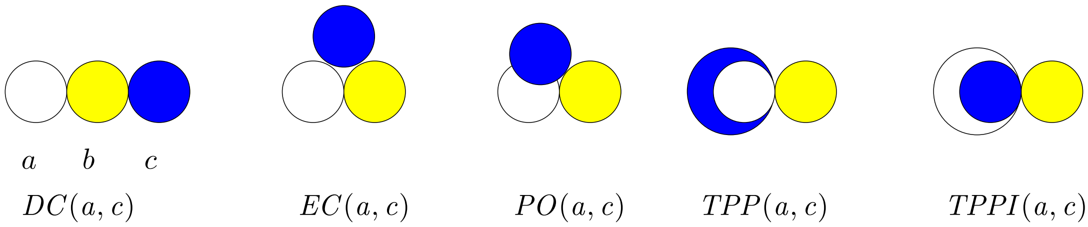


Figure 3: Illustration of the composition table entry for $EC(a, b) \circ EC(b, c)$.

inventory, in this case, by the coarseness of the RCC base-relationships.

A central role in many qualitative reasoning calculi is played by the so-called *composition table*. This table is used to solve the following basic inference problem: given the qualitative relationship between an object a and another object b, e.g. $S(a, b)$, and the qualitative relationship between b and yet another object c, e.g. $T(b, c)$, then what are the possible relationships between a and c? This question is answered by a lookup in the composition table, which just lists the possible relationships for $S \circ T$ (see below for the RCC composition tables). If we look up $contains \circ contains$, we should again get *contains* as the entry in the table, since *contains* is transitive. Generally, the entries in the composition table are disjunctions of base-relationships. For example, if we only know that a is adjacent to b, and b is adjacent to c, then the composition table tells us that the possible base-relationship between a and c must be one of the ones illustrated in Fig. 3. We assign the following semantics to a composition table entry: in FOPL axiomatics, each composition table entry represents a closed formulae of the form $\forall x, y, z : S(x, y) \land T(y, z) \to (R_1(x, z) \lor \ldots \lor R_n(x, z))$. Thus, the core idea is the following: in order to equip a DL with some kind of *composition table-based qualitative reasoning capabilities,* we need to supply (at least) *role axioms* of the form

$$\forall x, y, z : S(x, y) \land T(y, z) \to (R_1(x, z) \lor \ldots \lor R_n(x, z)),$$

or simply $S \circ T \sqsubseteq R_1 \sqcup \cdots \sqcup R_n$ in DL-syntax. A set of these role axioms if called a *role box.* A role axiom of the form $S \circ T \sqsubseteq R_1 \sqcup \cdots \sqcup R_n$ enforces $S^{\mathcal{I}} \circ T^{\mathcal{I}} \subseteq R_1^{\mathcal{I}} \cup \cdots \cup R_n^{\mathcal{I}}$ on the models $\mathcal{I}$. Returning to the previous example regarding *contains*, the role axiom should read $contains \circ contains \sqsubseteq contains$. Its FOPL translation $\forall x, y, z : contains(x, y) \land contains(y, z) \to contains(x, z)$ states that *contains* is transitively closed, which is what we want here: $contains^{\mathcal{I}} \circ contains^{\mathcal{I}} \subseteq contains^{\mathcal{I}}$. As already noted, the name of the RCC5 relationship for *contains* is PPI (proper part inverse; its inverse is called PP). The role box *derived from the RCC5 composition table* would, among others, contain the role axioms $PP \circ PP \sqsubseteq PP$ and $PPI \circ PPI \sqsubseteq PPI$.

We want to emphasize the use of *qualitative spatial relationships as roles* in a DL - we want to *quantify over roles corresponding to spatial relationships*, like in $\forall contains.\neg country$ ($\forall PPI.\neg country$). We believe that this is a first-order requirement for the qualitative spatial reasoning applications we are trying to realize; please see the subsequent deductive GIS example. Currently, only the logic $\mathcal{ALCRP}(\mathcal{S}_2)$ offers support for this kind of reasoning (see [15], [21], [20], [14]), but unfortunately somehow suffers from a severe syntax-restriction in order to achieve decidability.

1.3 Relationship to Previous Work

In previous work we investigated concept satisfiability of $\mathcal{ALC}$ extended with *arbitrary* role boxes containing role axioms of the form $S \circ T \sqsubseteq R_1 \sqcup \cdots \sqcup R_n$. More formally, a role axiom of the form $S \circ T \sqsubseteq R_1 \sqcup \cdots \sqcup R_n$ enforces $S^{\mathcal{I}} \circ T^{\mathcal{I}} \subseteq R_1^{\mathcal{I}} \cup \cdots \cup R_n^{\mathcal{I}}$ on the models $\mathcal{I}$. We did not require that the roles should be interpreted disjointly, and the resulting language was called $\mathcal{ALC}_{\mathcal{RA}^{\ominus}}$ (see [30]). Unfortunately, we have shown that concept satisfiability in $\mathcal{ALC}_{\mathcal{RA}^{\ominus}}$ (and even in smaller sub-languages) becomes undecidable. However, certain classes of so-called admissible role boxes satisfying additional conditions were shown to be decidable (e.g. the logic we called $\mathcal{ALC}_{\mathcal{RASG}}$, see [28]). Since role disjointness becomes an important requirement if one considers spatial reasoning applications, we investigated the logic $\mathcal{ALC}_{\mathcal{RA}}$ (see [29]), globally enforcing role disjointness on all roles, more formally: $R, S \in \mathcal{N_R}$, $R \neq S$: $R^{\mathcal{I}} \cap S^{\mathcal{I}} = \emptyset$. Unfortunately, $\mathcal{ALC}_{\mathcal{RA}}$ turned out to be undecidable as well (see [26], note that the $\mathcal{ALC}_{\mathcal{RA}^{\ominus}}$ proof could not be adapted to show undecidability of $\mathcal{ALC}_{\mathcal{RA}}$. Obviously, since $\mathcal{ALC}_{\mathcal{RA}}$ is already undecidable, if extension by inverse roles, called $\mathcal{ALCI}_{\mathcal{RA}}$, is undecidable as well.

However, undecidability arises in the *general* case. It was left open what happens if only *certain classes of role boxes* are considered, especially the role boxes which are obtained from translating the RCC composition tables. Analogously, the syntax-restriction criterion of $\mathcal{ALCRP}(\mathcal{S}_2)$ is the same as for $\mathcal{ALCRP}(\mathcal{D})$, and it was left open whether a more relaxed criterion could be defined for $\mathcal{ALCRP}(\mathcal{S}_2)$, which is an instantiation of $\mathcal{ALCRP}(\mathcal{D})$. Depending on the exploited RCC composition table, we will call the special $\mathcal{ALCI}_{\mathcal{RA}}$ specializations $\mathcal{ALCI}_{\mathcal{RCC}8}$, $\mathcal{ALCI}_{\mathcal{RCC}5}$, and so on. Concerning $\mathcal{ALCI}_{\mathcal{RCC}8}$, the question whether it might be decidable was basically already raised by Cohn in [8], where he suggested to use a pair of modal operators $\Box_R$ and $\Diamond_R$ for each available spatial base-relationship R of RCC. Subsequent work with Bennett (see [5, 6]) then focused on encoding of RCC relations and networks in (propositional) modal and intuitionistic logic. However, they did not investigate quantification over RCC relationships, as originally proposed by Cohn in [8] (e.g., it is impossible to express concepts like $\forall contains.\neg country$ in these logics).

It should be noted that the role axioms of the proposed form have a general-purpose status, and from this point of view, have a similar epistemic status like "$\sqcap$", "$\exists$" and "$\forall$", etc. To demonstrate the general-purpose status of $S \circ T \sqsubseteq R_1 \sqcup \cdots \sqcup R_n$, simply consider the role *has_niece* – axioms of the form $has\_brother \circ has\_daughter \sqsubseteq has\_niece$ as well as $has\_sister \circ has\_daughter \sqsubseteq has\_niece$ provide the most natural description of these role interrelationships. However, despite this general purpose status, our main motivation for the investigation was to support qualitative spatial reasoning in the described way (see also [19]).

2 The $\mathcal{ALCI}_{\mathcal{RCC}}$ Family

The considered family of $\mathcal{ALCI}_{\mathcal{RCC}}$ logics is defined as follows:

Definition 1 (Syntax of $\mathcal{ALCI}_{\mathcal{RCC}}$) Let $\mathcal{N}_{\mathcal{C}}$ and $\mathcal{N}_{\mathcal{R}}$ be two disjoint sets of symbols, called *concept names* (also called atomic concepts) and *role names* (also called atomic roles or simply roles).

The syntax is defined inductively, and is borrowed from $\mathcal{ALC}$ ($\mathcal{ALCI}$):

- Every concept name $C \in \mathcal{N}_{\mathcal{C}}$ is a concept.
- If R is a role from $\mathcal{N}_{\mathcal{R}}$, and C and D are concepts, then also $\neg C$, $C \sqcap D$, $C \sqcup D$, $\exists R.C$ and $\forall R.C$ are concepts (also called compound concepts; $\exists \mathsf{inv}(R).C$ and $\forall \mathsf{inv}(R).C$ are also possible, however, since $\mathcal{N}_{\mathcal{R}}$ is closed under applications of inv, it is not strictly necessary to define these as additional concepts here).

In the following, disjunctions of roles are written in curly brackets. If $R = \{S_1, \ldots, S_n\}$ is a disjunction of (disjunctive) roles S_i, we also write $\forall \{S_1, \ldots, S_n\}.C$ ($\exists \{S_1, \ldots, S_n\}.C$) as a shorthand for $\forall S_1.C \sqcap \ldots \sqcap \forall S_n.C$ ($\exists S_1.C \sqcup \ldots \sqcup \exists S_n.C$).

Two other important abbreviations are

- $\top =_{def} C \sqcup \neg C$ (True or "Verum"), and
- $\bot =_{def} C \sqcap \neg C$ (False or "Falsum"),

for some arbitrary $C \in \mathcal{N}_{\mathcal{C}}$.

We feel free to use various other standard boolean constructors which are easily definable as abbreviations as well, e.g. $(C \to D) =_{def} (\neg C \sqcup D)$, etc.

The set of *subconcepts* (subexpressions) of a concept C is denoted by $\mathsf{sub}(C)$.

According to the different sets of roles $\mathcal{N}_{\mathcal{R}}$ corresponding to different sets of RCC relationships, we define the following logics:

- $\mathcal{ALCI}_{\mathcal{RCC}8}$: $\mathcal{N}_{\mathcal{R}} = \{DC, EC, PO, EQ, TPP, TPPI, NTPP, NTPPI\}$.
- $\mathcal{ALCI}_{\mathcal{RCC}5}$: $\mathcal{N}_{\mathcal{R}} = \{DR, PO, EQ, PP, PPI\}$.
  In $\mathcal{ALCI}_{\mathcal{RCC}8}$, PP and PPI are definable as
  $PP =_{def} \{TPP, NTPP\}$ and $PPI =_{def} \{TPPI, NTPPI\}$.
- $\mathcal{ALCI}_{\mathcal{RCC}3}$: $\mathcal{N}_{\mathcal{R}} = \{DR, ONE, EQ\}$.
  In $\mathcal{ALC}_{\mathcal{RCC}5}$, ONE is definable as $ONE =_{def} \{PP, PPI, PO\}$.
- $\mathcal{ALCI}_{\mathcal{RCC}2}$: $\mathcal{N}_{\mathcal{R}} = \{DR, O\}$.
  In $\mathcal{ALCI}_{\mathcal{RCC}3}$, O is definable as $O =_{def} \{ONE, EQ\}$; and finally,
- $\mathcal{ALCI}_{\mathcal{RCC}1}$: $\mathcal{N}_{\mathcal{R}} = \{SR\}$.
  In $\mathcal{ALCI}_{\mathcal{RCC}2}$, SR is definable as $SR =_{def} \{DR, O\}$.

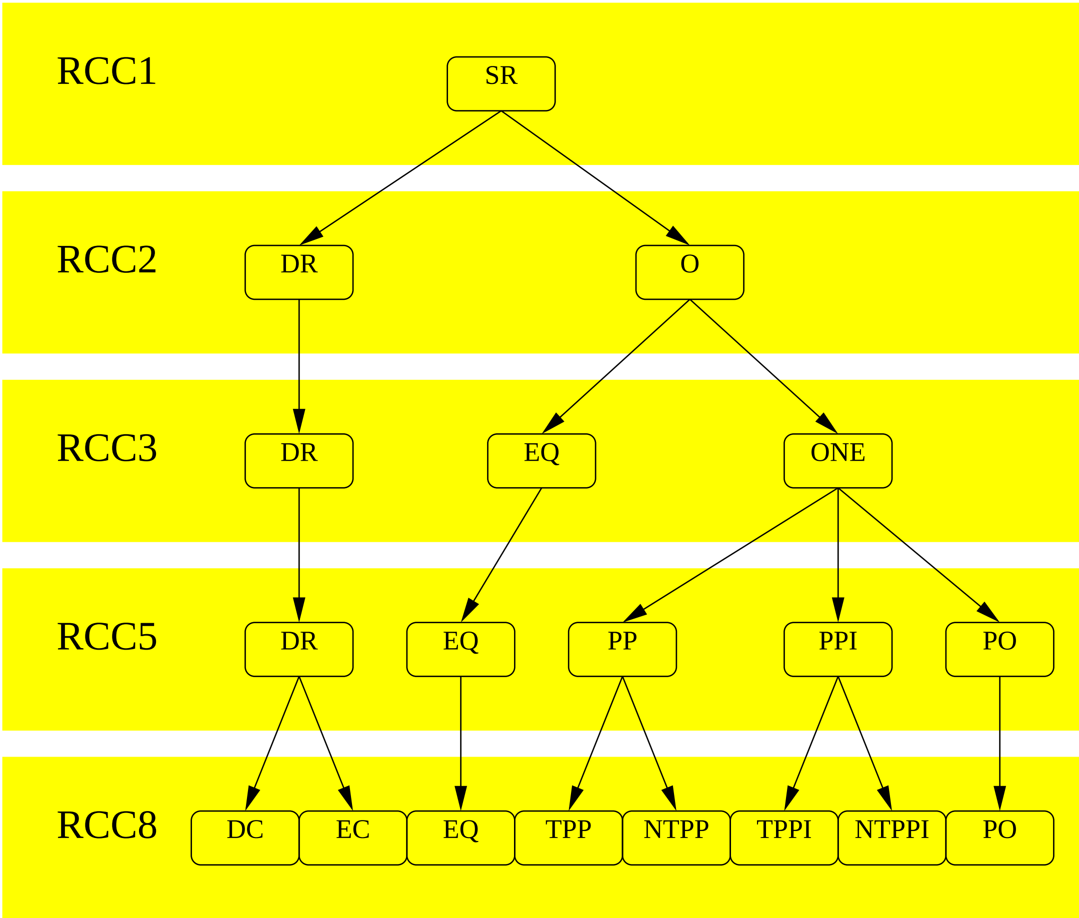


Figure 4: RCC relationships on various levels of granularity: the RCC5 relationships are obtained from the RCC8 relationships by collapsing DC and EC into DR; TPP and $NTPP$ into PP, as well as $TPPI$ and $NTPPI$ into PPI. In RCC3, $\{PP, PPI, PO\}$ are collapsed into ONE ("Overlapping but not equal"), and ONE, EQ into O in the case of RCC2 ("Overlapping"). The coarsest version – RCC1 – has only one relationship, called SR (for "Spatially Related").

It is a very distinguishing feature of the $\mathcal{ALCI}_{\mathcal{RCC}}$ logics that the set of role names $\mathcal{N_R}$ is *fixed* and *finite*. In contrast, the ordinary $\mathcal{ALCI}$ syntax just states that $\mathcal{N_R}$ is an arbitrary *countable* (possibly infinite) set.

We use the function inv to refer to the corresponding converse role (e.g. $PPI = \mathsf{inv}(PP)$, $DR = \mathsf{inv}(DR)$). Please note that inv is total on $\mathcal{N_R}$; and that $\mathcal{N_R}$ is closed under applications of inv (for all $R \in \mathcal{N_R}$, also $\mathsf{inv}(R) \in \mathcal{N_R}$).

These relationships shall exhibit the properties as given below. Some of these properties are consequences of the *composition tables* to be specified below, but others are just presupposed. The following properties should reflect the inherent spatial characteristics of the relationships; recall that a reflexive, symmetric and transitive relationship is called an equivalence relation, and that a irreflexive, asymmetric and transitive relationship is called a strict partial order (SPO):

- *RCC1:*
  - *SR* (*spatially related*): reflexive, symmetric, transitive
- *RCC2:*
  - *O* (*overlapping*): reflexive, symmetric
  - *DR* (*discrete*): symmetric, irreflexive
- *RCC3:*
  - *EQ* (*equal*): reflexive, symmetric, transitive
  - *ONE* (*overlapping, but not equal*): symmetric, irreflexive
  - *DR* (*discrete*): symmetric, irreflexive
- *RCC5:*
  - *EQ* (*equal*): reflexive, symmetric, transitive
  - *PO* (*partial overlapping*): symmetric, irreflexive
  - *DR* (*discrete*): symmetric, irreflexive
  - *PP* (*proper part*): asymmetric, irreflexive, transitive
  - *PPI* (*proper part inverse*): asymmetric, irreflexive, transitive
- *RCC8* has eight spatial roles; they are illustrated in Fig. 2. All roles but *EQ* are irreflexive, which is reflexive. All roles but *TPP*, *TPPI*, *NTPP*, *NTPPI* are symmetric, which are asymmetric. *TPP* and *TPPI* are *not* transitive, but *NTPP* and *NTPPI* are.

  In contrast to RCC5, RCC8 takes the relationship of an object to the *border* of an other object as a further distinguishing spatial characteristics into account. We can now differentiate whether two objects are *disjoint* or whether they are *touching.* Topologically speaking, in both cases their interiors do not share any common points. However, we can distinguish the two cases by observing that the *borders* of the two objects are intersecting in the touching case, which is not the case if the two objects are really disjoint from each other. The *DR*-relationship of RCC5 is therefore partioned into the relationships *EC* for *externally connected* and *DC*, *disconnected.* Considering the border of a *containing object* resp. the border of an object that has a *proper part*, i.e. of an object that has a *PPI* successor, we can now distinguish whether the contained object touches the border of its containing object or does not. In the former case, we say the contained object is *tangentially contained,* whereas in the latter case we say it is *non-tangentially contained.* The corresponding RCC8 relationships are called *TPPI* and *NTPPI*; they have corresponding inverse relationships *TPP* and *NTPP*.

This hierarchy of spatial relations is visualized in Figure 4; arrows indicate the subset relationship (the hierarchy is similar to the one discussed in [13]). Each relation is assumed to be *exactly equal* to the union of its immediate disjoint sub-relationships. The children relationships form a partition of the (coarser) parent relationship. For example, *PP* is partioned by *TPP* and *NTPP*.

The *role boxes* of $\mathcal{ALCI}_{\mathcal{RCC}1} \ldots \mathcal{ALCI}_{\mathcal{RCC}8}$ contain the role axioms that can be obtained from the following RCC composition tables (the RCC5 and RCC8 tables are taken from [22], [9], the others are defined by the author). For example, in the case of RCC1 we have the (trivial) role box $\{SR \circ SR \sqsubseteq SR\}$, according to the composition table entry for $SR \circ SR$ in the table given below, and so on for the other DLs. Please note that "$*$" always denotes the disjunction of all base-relations such that $* =_{def} \mathcal{N_R}$ for the appropriate set of role names $\mathcal{N_R}$ in that logic. Below the composition tables for RCC1, RCC2, RCC3, RCC5 and RCC8 are given:

∘	SR(a,b)
SR(b,c)	*

RCC1 Table

∘	DR(a,b)	O(a,b)
DR(b,c)	*	*
O(b,c)	*	*

RCC2 Table

∘	DR(a,b)	ONE(a,b)	EQ(a,b)
DR(b,c)	*	DR ONE	DR
ONE(b,c)	DR ONE	*	ONE
EQ(b,c)	DR	ONE	EQ

RCC3 Table

∘	DR(a,b)	PO(a,b)	EQ(a,b)	PPI(a,b)	PP(a,b)
DR(b,c)	*	DR PO PPI	DR	DR PO PPI	DR
PO(b,c)	DR PO PP	*	PO	PO PPI	DR PO PP
EQ(b,c)	DR	PO	EQ	PPI	PP
PP(b,c)	DR PO PP	PO PP	PP	PO EQ PP PPI	PP
PPI(b,c)	DR	DR PO PPI	PPI	PPI	*

RCC5 Table

$\circ$	DC(a,b)	EC(a,b)	PO(a,b)	TPP(a,b)	NTPP(a,b)	TPPI(a,b)	NTPPI(a,b)	EQ(a,b)
DC(b,c)	*	DC EC PO TPPI NTPPI	DC EC PO TPPI NTPPI	DC	DC	DC EC PO TPPI NTPPI	DC EC TPPI NTPPI	DC
EC(b,c)	DC EC PO TPP NTPP	DC EC PO TPP TPPI EQ	DC EC PO TPPI NTPPI	DC EC	DC	EC PO TPPI NTPPI	PO TPPI NTPPI	EC
PO(b,c)	DC EC PO TPP NTPP	DC EC PO TPP NTPP	*	DC EC PO TPP NTPP	DC EC PO TPP NTPP	PO TPPI NTPPI	PO TPPI NTPPI	PO
TPP(b,c)	DC EC PO TPP NTPP	EC PO TPP NTPP	PO TPP NTPP	TPP NTPP	NTPP	PO EQ TPP TPPI	PO TPPI NTPPI	TPP
NTPP(b,c)	DC EC PO TPP NTPP	PO TPP NTPP	PO TPP NTPP	NTPP	NTPP	PO TPP NTPP	PO TPPI TPP NTPP NTPPI EQ	NTPP
TPPI(b,c)	DC	DC EC	DC EC PO TPPI NTPPI	DC EC PO TPP TPPI EQ	DC EC PO TPP NTPP	TPPI NTPPI	NTPPI	TPPI
NTPPI(b,c)	DC	DC	DC EC PO TPPI NTPPI	DC EC PO TPPI NTPPI	*	NTPPI	NTPPI	NTPPI
EQ(b,c)	DC	EC	PO	TPP	NTPP	TPPI	NTPPI	EQ

RCC8 Table

Finally, to finish the definition of the syntax of the $\mathcal{ALCI}_{\mathcal{RCC}}$ family, we define the notions of *TBox*, *ABox*, and *Knowledge Base*:

A *TBox* as a finite set of axioms of the form $C \dot{\sqsubseteq} D$, where C and D are arbitrary $\mathcal{ALCI}_{\mathcal{RCC}}$ concepts (but concepts in the same $\mathcal{ALCI}_{\mathcal{RCC}}$ language of the family). Such an axiom is also called a *general concept inclusion axiom*, or *GCI*. We use $C \dot{\equiv} D$ as a shorthand for $\{C \dot{\sqsubseteq} D, D \dot{\sqsubseteq} C\}$.

An *ABox* $\mathfrak{A}$ is a finite set of *concept membership* and/or *role membership assertions:* if i, j are *ABox individuals* (from a set of *ABox individuals* $\mathcal{N}_{\mathcal{I}}$) and C is an $\mathcal{ALCI}_{\mathcal{RCC}}$ concept, and R is a role name, then $i : C$ is a *concept membership assertion (axiom)*, and $(i, j) : R$ is a *role membership assertion (axiom)*. Given an *ABox* $\mathfrak{A}$, we refer to the set of individuals used within $\mathfrak{A}$ as $\mathsf{individuals}(\mathfrak{A})$.

A *Knowledge Base* is simply a tuple $(\mathfrak{A}, \mathfrak{T})$, where $\mathfrak{A}$ is an ABox, and $\mathfrak{T}$ is a TBox. ■

An other important syntactic notion is the *Negation Normal Form*:

Definition 2 (Negation Normal Form of a Concept (NNF)) A concept C is said to be in *Negation Normal Form* (NNF), if the negation operator "$\neg$" only appears in front of *concept names.* Each concept can be brought into NNF by "pushing in" the negation sign, e.g. by exhaustively exploiting the equivalences $\neg\neg C \equiv C$, $\neg(C \sqcap D) \equiv (\neg C) \sqcup (\neg D)$, $\neg(C \sqcup D) \equiv (\neg C) \sqcap (\neg D)$, $\neg(\exists R.C) \equiv \forall R.\neg C$, $\neg(\forall R.C) \equiv \exists R.\neg C$. ■

Definition 3 (Semantics of the $\mathcal{ALCI}_{\mathcal{RCC}}$ Family) An *interpretation* $\mathcal{I} =_{def} (\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}})$ consists of a non-empty set $\Delta^{\mathcal{I}}$, called the domain of $\mathcal{I}$, and an interpretation function $\cdot^{\mathcal{I}}$, that maps every concept name C to a subset of $\Delta^{\mathcal{I}}$ ($C^{\mathcal{I}}$ is called the *extension* or interpretation of C), and every role name R from $\mathcal{N}_{\mathcal{R}}$ (of the considered member in the $\mathcal{ALCI}_{\mathcal{RCC}}$ family) to a subset of $\Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}}$ ($R^{\mathcal{I}}$ is called the extension or interpretation of R). It is sufficient if $\cdot^{\mathcal{I}}$ is defined only for concept names which are actually referenced; however, $\cdot^{\mathcal{I}}$ must be total on $\mathcal{N}_{\mathcal{R}}$. Elements from $\Delta^{\mathcal{I}}$ are called points, nodes or individuals.

A *(Kripke) frame* is an interpretation which only fixes the extensions of the role names (i.e., the domain of $\cdot^{\mathcal{I}}$ coincides with $\mathcal{N}_{\mathcal{R}}$). A *frame condition* is a semantic requirement that has to hold on the extensions of the role names. For example, a frame condition might state that $R^{\mathcal{I}}$ must be transitively closed - frames that satisfy this frame condition would be called transitive frames. We also say that a model or interpretation is *based on a frame* or based on frame from a class of frames (e.g., a frame from the class of transitive frames). These notions are adopted from modal logics and will be needed when we outline the relationships to modal logics.

In case we consider a non-empty ABox (or Knowledge Base), the interpretation function also has to supply a mapping from the set of ABox individual names $\mathcal{N}_{\mathcal{I}}$ into $\Delta^{\mathcal{I}}$; if we exploit the *unique name assumption*, this mapping has to be *injective.* It is sufficient if $\cdot^{\mathcal{I}}$ is defined solely for individual names which are actually referenced in the considered ABox.

Whenever $\langle i, j\rangle \in R^{\mathcal{I}}$ for some role R, we call j an *R-successor* (or simply *successor*) of i, and i an *R-predecessor* of j. We use these terms also if $i, j \in \mathsf{individuals}(\mathfrak{A})$ and $\langle i^{\mathcal{I}}, j^{\mathcal{I}}\rangle \in R^{\mathcal{I}}$, or if only $(i, j) : R \in \mathfrak{A}$.

Given an interpretation $\mathcal{I}$, every (possibly compound) concept C can uniquely be interpreted by using the following definitions (we write $X^{\mathcal{I}}$ instead of $\cdot^{\mathcal{I}}(X)$):

$$
\begin{aligned}
(\neg C)^{\mathcal{I}} &=_{def} \Delta^{\mathcal{I}} \setminus C^{\mathcal{I}} \\
(C \sqcap D)^{\mathcal{I}} &=_{def} C^{\mathcal{I}} \cap D^{\mathcal{I}} \\
(C \sqcup D)^{\mathcal{I}} &=_{def} C^{\mathcal{I}} \cup D^{\mathcal{I}} \\
(\exists R.C)^{\mathcal{I}} &=_{def} \{\, i \in \Delta^{\mathcal{I}} \mid \exists j \in C^{\mathcal{I}} : \langle i, j\rangle \in R^{\mathcal{I}} \,\} \\
(\forall R.C)^{\mathcal{I}} &=_{def} \{\, i \in \Delta^{\mathcal{I}} \mid \forall j : \langle i, j\rangle \in R^{\mathcal{I}} \rightarrow j \in C^{\mathcal{I}} \,\}
\end{aligned}
$$

It is therefore sufficient to provide the interpretations for the concept *names* and roles, since the extension $C^{\mathcal{I}}$ of every concept C is uniquely determined then.

W.r.t. the interpretations of the role names, we require that the following *frame conditions* must hold in $(\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}})$:

- *"One cluster"* requirement: $\forall x, y \in \Delta^{\mathcal{I}} : \langle x, y\rangle \in \bigcup_{R \in \mathcal{N}_{\mathcal{R}}} R^{\mathcal{I}}$
- *Converse* requirement: $R^{\mathcal{I}} = (\mathsf{inv}(S)^{\mathcal{I}})^{-1}$ iff $R = \mathsf{inv}(S)$
- *Role composition* requirements: $S^{\mathcal{I}} \circ T^{\mathcal{I}} \subseteq R_1^{\mathcal{I}} \cup \cdots \cup R_n^{\mathcal{I}}$ if $S \circ T \sqsubseteq R_1 \sqcup \cdots \sqcup R_n$ is an entry in the corresponding RCC composition table

- *Disjointness* requirement: for all $R, S \in \mathcal{N}_{\mathcal{R}}$ with $R \neq S$: $R^{\mathcal{I}} \cap S^{\mathcal{I}} = \emptyset$
- In case of $\mathcal{ALCI}_{\mathcal{RCC}3}$, $\mathcal{ALCI}_{\mathcal{RCC}5}$ and $\mathcal{ALCI}_{\mathcal{RCC}8}$, we distinguish *two kinds of semantics for the EQ relation:*
  - The *strong* EQ-semantics: $Id(\Delta^{\mathcal{I}}) = EQ^{\mathcal{I}}$
  - The *weak* EQ-semantics: $Id(\Delta^{\mathcal{I}}) \subseteq EQ^{\mathcal{I}}$

  $Id(\Delta^{\mathcal{I}}) =_{def} \{ <x, x> \mid x \in \Delta^{\mathcal{I}} \}$ is the *identity relation.* It depends on the application context which EQ-semantics is more appropriate.
- In case of $\mathcal{ALCI}_{\mathcal{RCC}1}$, $Id(\Delta^{\mathcal{I}}) \subseteq SR^{\mathcal{I}}$
- In case of $\mathcal{ALCI}_{\mathcal{RCC}2}$, $Id(\Delta^{\mathcal{I}}) \subseteq O^{\mathcal{I}}$

A *model* for a concept (ABox, TBox, ...) is an interpretation in which this concept (ABox, TBox, ...) is satisfied. Depending on the syntactic item, satisfiability is defined as follows:

- An interpretation $\mathcal{I}$ is a model of a concept C, written $\mathcal{I} \models C$, iff $C^{\mathcal{I}} \neq \emptyset$.
- An interpretation $\mathcal{I}$ is a model of a composition-based role inclusion axiom $S \circ T \sqsubseteq R_1 \sqcup \cdots \sqcup R_n \in \mathfrak{R}$, written $\mathcal{I} \models S \circ T \sqsubseteq R_1 \sqcup \cdots \sqcup R_n$, iff $S^{\mathcal{I}} \circ T^{\mathcal{I}} \subseteq R_1^{\mathcal{I}} \cup \cdots \cup R_n^{\mathcal{I}}$.
- An interpretation $\mathcal{I}$ is a model of a role box $\mathfrak{R}$, written $\mathcal{I} \models \mathfrak{R}$, iff for all role axioms $S \circ T \sqsubseteq R_1 \sqcup \cdots \sqcup R_n \in \mathfrak{R}$: $\mathcal{I} \models S \circ T \sqsubseteq R_1 \sqcup \cdots \sqcup R_n$.
- An interpretation $\mathcal{I}$ is a model of a GCI $C \dot{\sqsubseteq} D$, written $\mathcal{I} \models C \dot{\sqsubseteq} D$, iff $C^{\mathcal{I}} \subseteq D^{\mathcal{I}}$.
- An interpretation $\mathcal{I}$ is a model of a TBox $\mathfrak{T}$, written $\mathcal{I} \models \mathfrak{T}$, iff for all GCIs $C \dot{\sqsubseteq} D \in \mathfrak{R}$: $\mathcal{I} \models C \dot{\sqsubseteq} D$.
- An interpretation $\mathcal{I}$ is a model of a concept membership assertion $i : C$, written $\mathcal{I} \models i : C$, iff $i^{\mathcal{I}} \in C^{\mathcal{I}}$.
- An interpretation $\mathcal{I}$ is a model of a role membership assertion $(i, j) : R$, written $\mathcal{I} \models (i, j) : R$, iff $<i^{\mathcal{I}}, j^{\mathcal{I}}> \in R^{\mathcal{I}}$.
- An interpretation $\mathcal{I}$ is a model of an ABox $\mathfrak{A}$, written $\mathcal{I} \models \mathfrak{A}$, iff for all ABox axioms $axiom \in \mathfrak{R}$: $\mathcal{I} \models axiom$. ■

The following basic observations are direct consequences of the semantics as given above:

1. $\mathcal{ALCI}_{\mathcal{RCC}}$ does not have the *tree model property:* due to the *one cluster requirement* the models have the form of complete graphs, K_n's (see Figure 1 for an illustration of how a typical model looks like).

2. PP (resp. PPI) is a transitively closed role ($PP^{\mathcal{I}} = (PP^{\mathcal{I}})^+$), due to $PP \circ PP \sqsubseteq PP$. This is a consequence of the RCC5 composition table.

3. Since $PP^{\mathcal{I}} \cap PPI^{\mathcal{I}} = \emptyset$ and $EQ^{\mathcal{I}} \cap PP^{\mathcal{I}} = \emptyset$, $PP^{\mathcal{I}}$ is also *irreflexive*, as well as *antisymmetric* (therefore *asymmetric*) - in short: a *strict partial order (SPO).*

4. $\mathcal{ALCI}_{\mathcal{RCC}5}$ and $\mathcal{ALCI}_{\mathcal{RCC}8}$ do *not* have the *finite model property*: $\exists PP.\top \sqcap \forall PP.\exists PP.\top$ has only infinite models. We therefore even have to deal with K_ω's (infinite complete graphs).

5. Models need not be *dense w.r.t.* PP/PPI, since only $PP \circ PP \sqsubseteq PP$, but *not* $PP \sqsubseteq PP \circ PP$.

6. Even though the *weak EQ-semantics* only requires that $EQ^{\mathcal{I}}$ is a superset of the identity relation Id, it is a consequence of the frame conditions that $EQ^{\mathcal{I}}$ is in fact a *congruence relation* for the roles (needless to say, Id is one as well). *The weak EQ-semantics can be made strong:* add, for all "relevant" concept names $CN \in \mathcal{N}_\mathcal{C}$, the TBox axiom $CN \sqsubseteq \forall EQ.CN$ to the original knowledge base (we will show later that knowledge base consistency can always be reduced to concept satisfiability). The nodes participating in an EQ clique then have equivalent theories which means their cliques can be collapsed into a single reflexive node, resulting in a model under the strong EQ-semantics.

7. In each logic of the $\mathcal{ALCI}_{\mathcal{RCC}}$ family we have a *universal role* $R_* =_{def} \mathcal{N}_\mathcal{R}$ (the disjunction of all roles, which refers to all other points in the model, *including* the point itself). This is due to the fact that models are *connected.* It is well-known that a universal role can be used to encode whole TBoxes into single concept expressions by a process called *internalization* (see below).

8. Considering the *strong EQ-semantics*, we also have a *difference role* $R_D =_{def} \mathcal{N}_\mathcal{R} \setminus \{EQ\}$ (referring to all other points in the model, *excluding* the point itself). It is well-known that a *difference role* has the expressive power to allow for the encoding of so-called *nominals* (nominals are concept names which are interpreted as singletons, therefore representing a single individual in $\Delta^{\mathcal{I}}$- we will exploit nominals in order to translate an $\mathcal{ALCI}_{\mathcal{RCC}}$ ABox into a single concept expression, see below).

So far for the immediate consequences from the semantics. Now for the reasoning tasks:

Definition 4 (Reasoning Problems) The following reasoning problems are also called *standard DL reasoning problems.* Since we do not exploit ABox reasoning in the following, we skipped some of the standard ABox reasoning problems.

- Given a concept C, the *concept satisfiability problem* is to check whether C has a model, i.e. whether there is some $\mathcal{I}$ such that $\mathcal{I} \models C$.

- Given two concepts C and D, the *subsumption problem* is to check whether $C^{\mathcal{I}} \subseteq D^{\mathcal{I}}$ holds for all $\mathcal{I}$. The subsumption relationship between concepts is denoted by $C \sqsubseteq D$. We might also say that C entails D ($C \models D$): whenever $\mathcal{I} \models C$, also $\mathcal{I} \models D$.

- Given a TBox $\mathfrak{T}$, the *TBox satisfiability problem* it to check whether a TBox admits a model, i.e. whether there is some $\mathcal{I}$ such that $\mathcal{I} \models \mathfrak{T}$.

- Given a TBox $\mathfrak{T}$ and a concept C, the *concept satisfiability problem w.r.t. a TBox* is to determine whether there is model $\mathcal{I}$ such that $\mathcal{I} \models \mathfrak{T}$, $\mathcal{I} \models C$.

- Given an ABox $\mathfrak{A}$, the *ABox satisfiability problem* is to determine whether there is a model $\mathcal{I}$ such that $\mathcal{I} \models \mathfrak{A}$.

- Given a Knowledge Base $(\mathfrak{T}, \mathfrak{A})$, the *Knowledge Base satisfiability problem* is to determine whether there is a model $\mathcal{I}$ such that $\mathcal{I} \models \mathfrak{A}$ as well as $\mathcal{I} \models \mathfrak{T}$. ■

Considering the availability of the difference role R_D in all super logics of $\mathcal{ALCI}_{\mathcal{RCC}3}$ adhering to the strong EQ-semantics, we see that we can encode *nominals* in our DLs:

Fact 1 (Availability of "Nominals") Let $\boxed{\text{i}} \in \mathcal{N_C}$ be a concept name. Consider the concept $\boxed{\text{i}} \sqcap \forall R_D.\neg\boxed{\text{i}}$. Note that due to the strong EQ-semantics $R_D =_{def} \mathcal{N_R} \setminus \{EQ\}$ is the so-called *difference role.* Inspecting the models of $\boxed{\text{i}} \sqcap \forall R_D.\neg\boxed{\text{i}}$, it becomes obvious that $\boxed{\text{i}}^{\mathcal{I}}$ must be a singleton. W.l.o.g. we may assume that $\boxed{\text{i}}^{\mathcal{I}} = \{i\}$, for an $i \in \Delta^{\mathcal{I}}$: $\boxed{\text{i}}$ is a *nominal.* Note that the difference role R_D is only definable if we employ the strong EQ-semantics, and that at least $\mathcal{ALCI}_{\mathcal{RCC}3}$ is needed. ■

Proof 1 Let $i \in (\boxed{\text{i}} \sqcap \forall R_D.\neg\boxed{\text{i}})^{\mathcal{I}}$. Assume that $\{i, j\} \subseteq \boxed{\text{i}}^{\mathcal{I}}$, with $i \neq j$. Since $i \neq j$ and due to the strong EQ-semantics, $\langle i, j\rangle \notin EQ^{\mathcal{I}}$, but $\langle i, j\rangle \in R_*^{\mathcal{I}}$. Thus $\langle i, j\rangle \in R_D^{\mathcal{I}}$, since $R_D =_{def} R_* \setminus \{EQ\}$. Since $i \in (\forall R_D.\neg\boxed{\text{i}})^{\mathcal{I}}$ we also have $j \in (\neg\boxed{\text{i}})^{\mathcal{I}}$, contradicting the assumption that $j \in \boxed{\text{i}}^{\mathcal{I}}$. □

Given the availability of nominals, we can do interesting things:

- We can limit the *maximal cardinality* of the models - suppose we are only interested in models having at most n points. Let *marker* $=_{def} \{\boxed{a_1}, \ldots, \boxed{a_n}\}$ with *marker* $\subseteq \mathcal{N_C}$ be a set of fresh concept names that do not yet appear in the knowledge base $\mathcal{K}$ under consideration. Then, by adding a *cover axiom* of the form $\top \sqsubseteq \boxed{a_1} \sqcup \cdots \sqcup \boxed{a_n}$ and a set of *disjointness axioms* of the form $\forall i \in 1 \ldots n : \boxed{a_i} \sqsubseteq \bigsqcap_{\boxed{a_j} \in marker \setminus \boxed{a_i}} \neg\boxed{a_j}$ as well as the axioms $\forall i \in 1 \ldots n : \boxed{a_i} \sqsubseteq \forall R_D.\neg\boxed{a_i}$ we will have enforced *finite models of maximal cardinality n.*

- We can translate whole ABoxes into concept expressions.

- We can even translate whole knowledge bases into concept expressions.

- For every inference task that references an ABox, we can switch to a *closed domain reasoning mode:* given an ABox $\mathfrak{A}$ referring to the individuals $\{i_1, i_2, \ldots, i_n\}$ (i.e., $\mathsf{individuals}(\mathfrak{A}) = \{i_1, i_2, \ldots, i_n\}$), we can arrange for closed domain reasoning such that every $\mathcal{I} = (\cdot^{\mathcal{I}}, \Delta^{\mathcal{I}})$ with $\mathcal{I} \models \mathfrak{A}$ also satisfies $\{i_1^{\mathcal{I}}, i_2^{\mathcal{I}}, \ldots, i_n^{\mathcal{I}}\} = \Delta^{\mathcal{I}}$. The encoding exploits nominals as well as cover axioms, and is similar to the construction given above, see below for the details. However, traditionally, all DL inference problems rely on the *open domain assumption*; in this case, only $\{i_1^{\mathcal{I}}, i_2^{\mathcal{I}}, \ldots, i_n^{\mathcal{I}}\} \subseteq \Delta^{\mathcal{I}}$ would hold, meaning that there might be individuals in $\Delta^{\mathcal{I}}$ which are not also explicitly mentioned in the ABox $\mathfrak{A}$. However, especially in our deductive GIS scenario, the closed domain assumption might be very useful and is also often considered in (more traditional) database applications of DLs.

Interestingly, all the mentioned reasoning problems can be reduced to the *concept satisfiability problem,* which is the central decision problem in any logic:[3] We also show how to translate an ABox and a TBox (thus, a whole knowledge base) into a single concept expression. We do so using nominals and internalization:

Fact 2 (Reduction to Concept Satisfiability) We show how to reduce the various satisfiability problems to the concept satisfiability problem:

- $C \sqsubseteq D$ iff $C \sqcap \neg D$ is unsatisfiable.

- The TBox $\mathfrak{T}$ is satisfiable iff the concept
  $$\forall R_*.\textstyle\bigsqcap_{C \dot\sqsubseteq D \in \mathfrak{T}} (\neg C \sqcup D)$$
  is satisfiable. This concept is also called the *internalization* of $\mathfrak{T}$. Note that $EQ \in R_*$, and the R_* is the *universal role* $R_* =_{def} \mathcal{N_R}$. The $\mathcal{ALCI}_{\mathcal{RCC}}$ family allows for *internalization of TBoxes.*

- The concept E is satisfiable *w.r.t. the TBox* $\mathfrak{T}$ iff the concept
  $$E \sqcap \forall R_*.\textstyle\bigsqcap_{C \dot\sqsubseteq D \in \mathfrak{T}} (\neg C \sqcup D)$$
  is satisfiable (w.r.t. the empty TBox).

- Given a knowledge base $\mathcal{K} = (\mathfrak{T}, \mathfrak{A})$ we show how to encode $\mathcal{K}$ into a *single concept term* $C_{\mathcal{K}}$, such that $\mathcal{K}$ and $C_{\mathcal{K}}$ are equi-satisfiable:

  Let $\mathcal{K} = (\mathfrak{A}, \mathfrak{T})$ be a knowledge base. To construct $C_{\mathcal{K}}$ from $\mathcal{K}$, we proceed as follows:

  1. $\mathfrak{A}' =_{def} \{\, i : \boxed{i} \mid i \in \mathsf{individuals}(\mathfrak{A}) \,\} \cup \mathfrak{A}$, where $\boxed{i}$ is fresh concept name which does not yet appear in $\mathcal{K}$.
  2. $\mathfrak{A}'' =_{def} \{\, (i,j) : R \mid (j,i) : \mathsf{inv}(R) \in \mathfrak{A}' \,\} \cup \mathfrak{A}'$.
  3. $\mathfrak{A}''' =_{def} \{\, (i,j) : * \mid \nexists R \in \mathcal{N_R} : (i,j) : R \in \mathfrak{A}'' \,\} \cup \mathfrak{A}''$; "*" is just a marker symbol with $* \notin \mathcal{N_R}$ denoting that no other edge between i, j is present.

[3] If the concept satisfiability problem is undecidable, then *the logic* is called undecidable.

4. Select a starting individual x_1 in $\mathfrak{A}'''$ and start *building a closed path p in* $\mathfrak{A}'''$ by traversing the role member ship assertions / edges in $\mathfrak{A}''$ in such a way that every individual and every edge is *at least visited once*; it is okay to visit edges and nodes more than once. Obviously, such a path always exists. Each time we traverse an edge we record the corresponding role membership assertion axiom in the path. An assembled path looks like this: $[R_1(x_1, x_2), R_2(x_2, x_3), \ldots R_n(x_n, x_n 1)]$.
5. Given a path p, we define (in Prolog-like notation) the translate-function:
   (a) $\mathsf{concept}(i) =_{def} \boxed{i} \sqcap \bigsqcap_{i:C \in \mathfrak{A}'''} C$
   (b) for the empty path "[]": $\mathsf{translate}([]) =_{def} \top$
   (c) for $R \neq *$: $\mathsf{translate}([R(i,j)|Rest]) =_{def}$
   $\mathsf{concept}(i) \sqcap \exists R.(\mathsf{concept}(j) \sqcap \mathsf{translate}(Rest))$
   (d) for $R = *$: $\mathsf{translate}([*(i,j)|Rest]) =_{def}$
   $\mathsf{concept}(i) \sqcap \exists R_*.(\mathsf{concept}(j) \sqcap \mathsf{translate}(Rest))$
6. $\mathfrak{T}' =_{def} \{ \boxed{i} \sqsubseteq \bigsqcap_{j \in \mathsf{individuals}(\mathfrak{A}''') \setminus i} \neg \boxed{j} \mid \ i \in \mathsf{individuals}(\mathfrak{A}''') \} \ \cup \mathfrak{T}$
7. $\mathfrak{T}'' =_{def} \{ \boxed{i} \sqsubseteq \forall R_D. \neg \boxed{i} \mid \ i \in \mathsf{individuals}(\mathfrak{A}''') \} \ \cup \mathfrak{T}'$
8. if we want *closed domain reasoning,* we additionally add the axiom $\top \sqsubseteq \bigsqcup_{i \in \mathsf{individuals}(\mathfrak{A}''')} \boxed{i}$ to $\mathfrak{T}''$
9. Finally, "internalize" the TBox $\mathfrak{T}''$:
   $C_{\mathfrak{T}''} =_{def} \forall R_*. \bigsqcap_{C \sqsubseteq D \in \mathfrak{T}''} (\neg C \sqcup D)$
10. Using the path p from above, the final concept term is then
   $C_{\mathcal{K}} =_{def} \mathsf{translate}(p) \sqcap C_{\mathfrak{T}''}$

It is easy to show that $\mathcal{K} = (\mathfrak{A}, \mathfrak{T})$ is satisfiable (under the closed domain assumption) iff $C_{\mathcal{K}}$ is satisfiable (under the closed domain assumption): given $\mathcal{I} \models (\mathfrak{A}, \mathfrak{T})$ we construct $\mathcal{I}' \models C_{\mathcal{K}}$ by assigning for all $i \in \mathsf{individuals}(\mathfrak{A})$ $\boxed{i}^{\mathcal{I}'} =_{def} \{i^{\mathcal{I}}\}$, and likewise the other way around. Note that the $\boxed{\text{i}}$'s are in fact nominals, one for each present ABox individual. ■

We have shown that once the concept satisfiability problem is solved, one can even decide $\mathcal{ALCI}_{\mathcal{RCC}}$ knowledge base consistency.

2.1 Does Role Disjointness Affect Concept Satisfiability?

How important ist the *JEPD*-property? Does it even make a difference w.r.t. concept satisfiability to state that roles should be interpreted as disjoint? For many DLs, role disjointness does not effect concept satisfiability. For example, it is well-known that the DL $\mathcal{ALC}$ has the *finite tree model property* - each satisfiable $\mathcal{ALC}$ concept has a model that has the form of a finite tree. This is not to say that there are not also other models of this concept that do not have the form of finite trees (e.g., might be infinite, might not be trees, or even both). However, since $\mathcal{ALC}$ has the finite tree model property, a role disjointness requirement doesn't make sense and in

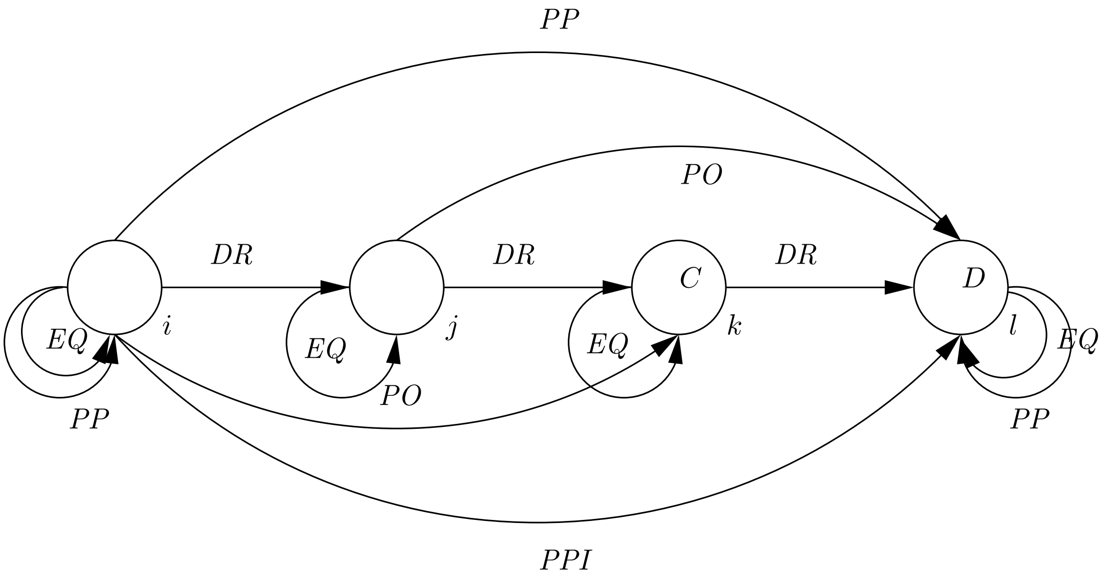


Figure 5: Example model under the weakened semantics assumption.

fact wouldn't affect satisfiability, since trees automatically satisfy the role disjointness requirement.

However, things are different for the $\mathcal{ALCI}_{\mathcal{RCC}}$ family. Obviously, $\mathcal{ALCI}_{\mathcal{RCC}}$ models are not trees – since the roles are *jointly exhaustive*, each individual is connected to every other via at least one role. The *pairwise disjoint* property ensures that "at least one role" means in fact *exactly* one role. The disjointness property is also responsible for destroying the *finite model property:* consider the $\mathcal{ALCI}_{\mathcal{RCC}5}$ concept

$$(\exists PP.\top) \sqcap (\forall PP.\exists PP.\top)$$

Recall that PP is transitively closed. Each model of this concept must contain an infinite chain of PP-successors, and it is easy to see that this infinite chain cannot be collapsed into a single reflexive node, since for each model $Id(\Delta^{\mathcal{I}}) \subseteq EQ^{\mathcal{I}}$ holds. Thus, a node i with $\lessdot i, i \gtrdot \in PP^{\mathcal{I}}$ would violate the role disjointness criterion, since already $\lessdot i, i \gtrdot \in EQ^{\mathcal{I}}$ holds.

Suppose for the moment that, semantically, we no longer require that PP must be irreflexive, that roles no longer must be interpreted as disjoint, but instead we *only* require that the role axioms must be satisfied such that for each $S \circ T \sqsubseteq R_1 \sqcup \cdots \sqcup R_n \in \mathfrak{R}$, $S^{\mathcal{I}} \circ T^{\mathcal{I}} \subseteq R_1^{\mathcal{I}} \cup \cdots \cup R_n^{\mathcal{I}}$ holds, as usual. Let's call this the *weakened $\mathcal{ALCI}_{\mathcal{RCC}}$ semantics.* Obviously, a broader class of models is now considered as appropriate. For example, $(\exists PP.\top) \sqcap (\forall PP.\exists PP.\top)$ would now not only have infinite models, but also *finite models, containing at least one reflexive point i* with $\lessdot i, i \gtrdot \in PP^{\mathcal{I}} \cap EQ^{\mathcal{I}}$, since the constraints on the interpretations of the roles are more relaxed, allowing a broader class of models, as described.

Then, a natural question is: are both notions of satisfiability equivalent in the sense that any concept that is satisfiable w.r.t. the original semantics is also satisfiable w.r.t. the weakened $\mathcal{ALCI}_{\mathcal{RCC}}$ semantics, and vice versa? The answer is negative: consider the $\mathcal{ALCI}_{\mathcal{RCC}5}$ concept

$$\begin{aligned}
&(\forall DR.\neg C) \sqcap (\forall PP.\neg C) \sqcap (\forall PPI.\neg C) \sqcap (\forall EQ.\neg C) \sqcap \\
&(\exists DR.(\ \ (\exists DR.(C \sqcap (\exists DR.D))) \sqcap \\
&\qquad\qquad (\forall DR.\neg D) \sqcap (\forall PP.\neg D) \sqcap (\forall PPI.\neg D) \sqcap (\forall EQ.\neg D)) \sqcap \\
&(\forall DR.\neg D) \sqcap (\forall PO.\neg D)
\end{aligned}$$

The concept is unsatisfiable in $\mathcal{ALCI}_{\mathcal{RCC}5}$, but under the weakened $\mathcal{ALCI}_{\mathcal{RCC}}$ semantics, the concept is satisfiable: an example model is given in Figure 5. Please note that $\lhd i, i \rhd \in PP^{\mathcal{I}} \cap PPI^{\mathcal{I}}$, $\lhd l, l \rhd \in PP^{\mathcal{I}} \cap PPI^{\mathcal{I}}$, $\lhd i, l \rhd \in PP^{\mathcal{I}} \cap PPI^{\mathcal{I}}$, $\lhd l, i \rhd \in PP^{\mathcal{I}} \cap PPI^{\mathcal{I}}$. Obviously, the role disjointness requirement ist not fulfilled, but the role box axioms are satisfied in this model.

It can be claimed that, from a modeling point of view, the *JEPD*-property and thus the additional frame conditions are of importance, since the notion of satisfiability would not make much sense otherwise. Obscure models (like the given one) could not be ruled out, admitting models with nodes being proper part of themselves, etc.

3 An $\mathcal{ALCI}_{\mathcal{RCC}5}$ Application: Reasoning in a Deductive Geographic Information System

Let us demonstrate by means of an example that even a restricted DL such as $\mathcal{ALCI}_{\mathcal{RCC}5}$ can be of value to support interesting spatial reasoning applications. Imagine a Geographic Information System (GIS) that does not only provide access to digital vector and/or raster maps (a GIS is sometimes simply defined as a "collection of digital maps"), but parallely also maintains a qualitative symbolic representation of the stored data, for example in forms of an RCC5 network.

Such a *hybrid representation schema* might have a number of advantages over conventional schemas: on one hand, the RCC network could serve as an index to actual stored geographic objects. Imagine a user wants to retrieve all german cities. In this case, the RCC network could be of great value: select the node that represents Germany, follow the *PPI*-links from there and collect all objects that are known to be *cities*. Computationally, this is much faster than computing the present containment relationsships on the fly using spatial indexing techniques. One the other hand, such a symbolic qualitative representation is a prerequisite if one wants to support deduction and reasoning on a logical level. For example, the consistency of the GIS could be checked automatically. See also [21] for interesting GIS-applications of DLs.

Assume the following TBox is used to model the geographic background knowledge of the system, as well as some specific geographic objects, like the rivers *Elbe* and *Alster* and the city of *Hamburg*. These are modeled as "individual" concepts here (however, a representation as ABox individuals would be more appropriate):

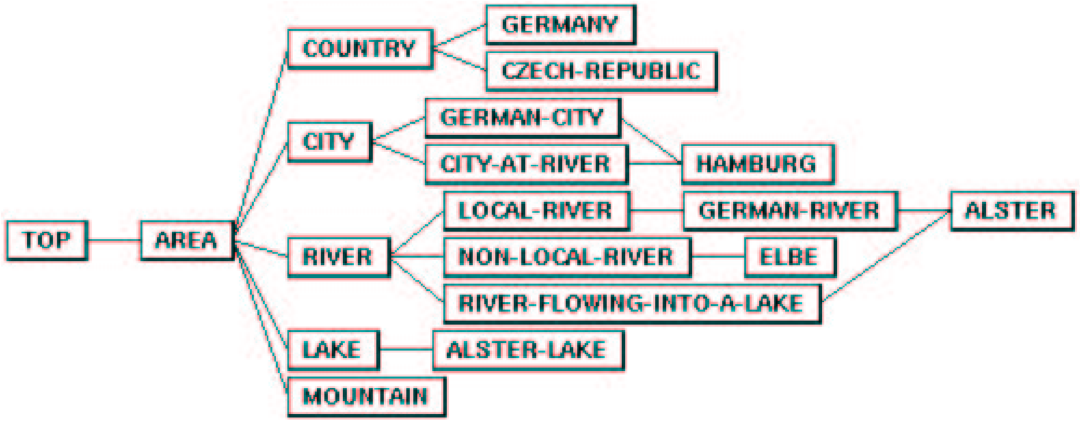


Figure 6: Computed taxonomy of the example TBox.

$$
\begin{array}{rcl}
country & \dot{\sqsubseteq} & area \\
city & \dot{\sqsubseteq} & area \\
river & \dot{\sqsubseteq} & area \\
lake & \dot{\sqsubseteq} & area \\
mountain & \dot{\sqsubseteq} & area \\
germany & \dot{\sqsubseteq} & country \\
czech\_republic & \dot{\sqsubseteq} & country \\
local\_river & \doteq & (river \sqcap \neg(\exists PO.country)) \\
non\_local\_river & \doteq & (river \sqcap (\exists PO.country)) \\
river\_flowing\_into\_a\_lake & \doteq & (river \sqcap (\exists PO.lake)) \\
german\_river & \doteq & (river \sqcap (\exists PP.germany) \sqcap (\forall PO.\neg country)) \\
german\_city & \doteq & (city \sqcap (\forall PP.country \rightarrow germany)) \\
city\_at\_river & \doteq & (city \sqcap (\exists PO.river)) \\
elbe & \dot{\sqsubseteq} & (river \sqcap (\exists PO.czech\_republic) \sqcap (\exists PO.germany)) \\
alster\_lake & \dot{\sqsubseteq} & lake \\
alster & \dot{\sqsubseteq} & (river \sqcap (\exists PP.germany) \sqcap (\exists PO.alster\_lake) \sqcap \\
 & & (\forall PO.\neg country) \sqcap (\forall PP.country \rightarrow germany)) \\
hamburg & \dot{\sqsubseteq} & (city \sqcap (\exists PO.alster))
\end{array}
$$

A DL-based GIS would automatically compute the subsumption hierarchy (taxonomy) as shown in Figure 6. Please note that most of the subsumption relationships in the taxonomy are not explicitly stated as TBox axioms. Their presence has been deduced by the system. For example, the system has deduced that *hamburg* is in fact a *german_city*, and also a *city_at_a_river*. The latter inference is not too surprising since it is explictly stated in the TBox that

$$hamburg \dot{\sqsubseteq} (city \sqcap (\exists PO.alster)),$$

and *alster* is known to be a *river*, but the former inference that *hamburg* must be a *german_city* is quite remarkable. The involved reasoning process goes like that:

- We want to show that $hamburg \sqsubseteq germany\_city$. To do this, we show that $hamburg \sqcap \neg germany\_city$ is unsatisfiable, which means that

  $$(city \sqcap (\exists PO.alster)) \sqcap \neg(city \sqcap (\forall PP.country \rightarrow germany))$$

  must be unsatisfiable, which is equivalent to

  $$(city \sqcap (\exists PO.alster)) \sqcap (\neg city \sqcup (\exists PP.country \sqcap \neg germany)).$$

  Since $city \ldots \sqcap \neg city$ is an obvious contradition, we only have to show that

  $$(city \sqcap (\exists PO.alster)) \sqcap (\exists PP.country \sqcap \neg germany)$$

  is unsatisfiable.

- Plugging in the definition of $alster$ we get

  $$\begin{aligned}&(city \sqcap (\exists PO.\ (river \sqcap (\exists PP.germany) \sqcap (\exists PO.alster\_lake) \sqcap \\ &\qquad (\forall PO.\neg country) \sqcap (\forall PP.country \rightarrow germany))) \sqcap \\ &(\exists PP.country \sqcap \neg germany)).\end{aligned}$$

  Let's call this concept X, and suppose there is a $city$ for which the whole expression holds, such that $city \in X^{\mathcal{I}}$. Due to $city \in (\exists PP.country \sqcap \neg germany)^{\mathcal{I}}$ (this is a conjunct of X) there must exist an other node, call it $country \sqcap \neg germany$, such that $<city, (country \sqcap \neg germany)> \in PP^{\mathcal{I}}$. Due to $city \in (\exists PO.alster)^{\mathcal{I}}$ there is yet an other node, call it $alster$, such that $<city, alster> \in PO^{\mathcal{I}}$ holds. Due to $<(country \sqcap \neg germany), city> \in PPI^{\mathcal{I}}$ and $PPI \circ PO \sqsubseteq PPI \sqcup PO$, we either have $<(country \sqcap \neg germany), alster> \in PPI^{\mathcal{I}}$ or $<(country \sqcap \neg germany), alster> \in PO^{\mathcal{I}}$. In the first case, $<alster, (country \sqcap \neg germany)> \in PP^{\mathcal{I}}$ yields an immediate contradiction, since $alster \in (\forall PP.country \rightarrow germany^{\mathcal{I}})$. In the second case, we get a contradiction due to $alster \in (\forall PO.\neg country)^{\mathcal{I}}$. Summing up, we have shown that $hamburg \sqcap \neg germany\_city$ is unsatisfiable, which means that $hamburg \sqsubseteq germany\_city$ must hold.

It should be noted that the subsumption hierarchy in Figure 6 was actually computed by a working prototype system of the deductive GIS. Unfortunately, we do not know yet whether $\mathcal{ALCI}_{\mathcal{RCC}5}$ is decidable (see below). Nevertheless, the underlying tableau calculus of the prototype can be shown to be sound and complete.

4 First Results Concerning the $\mathcal{ALCI}_{\mathcal{RCC}}$ Family

In the following we survey what we have found out so far about the $\mathcal{ALCI}_{\mathcal{RCC}}$ family. The main result of the section is the proof of decidability of $\mathcal{ALCI}_{\mathcal{RCC}3}$, which also implies decidability of $\mathcal{ALCI}_{\mathcal{RCC}1}$ and $\mathcal{ALCI}_{\mathcal{RCC}3}$. However, we will make some remarks on $\mathcal{ALCI}_{\mathcal{RCC}1}$ and $\mathcal{ALCI}_{\mathcal{RCC}2}$. We will also highlight the relationsships to modal logics. Open problems are discussed.

4.1 $\mathcal{ALCI}_{\mathcal{RCC}1}$ is Decidable

$\mathcal{ALCI}_{\mathcal{RCC}1}$ is decidable and equivalent to the modal logic "S5" (see [7], [10]). We might even say that it is a syntactic variant of "S5": simply replace $\forall SR.C$ with $\Box C$,

$\exists SR.C$ with $\Diamond C$, "$\sqcap$" with "$\wedge$", "$\sqcup$" with "$\vee$", etc.

"S5" is the well-known normal modal logic where the accessibility relation is interpreted as an *equivalence relation.* Informally, we may say that "S5" models are complete, undirected and reflexive graphs, with nodes being labeled by propositional information (concept names). It is well-known that "S5" is NP-complete.

The very special structure of the models gives rise to the following characteristic validities, which are also known as *reduction principles.* Let C be an arbitrary concept resp. "S5"-formula. Then, the following equivalences hold (among others, of course):

- $\Diamond C \equiv \Box\Diamond C$ (equivalently, $\exists SR.C \leftrightarrow \forall SR.\exists SR.C$ for all concepts C),
- $\Box C \equiv \Diamond\Box C$,
- $\Diamond C \equiv \Diamond\Diamond C$,
- $\Box C \equiv \Box\Box C$

Please note that the validity of these axioms can be easily seen: suppose some node in a model has a successor where "C" holds. This node would satisfy $\Diamond C$. Suppose this node has other different successors as well. Take an arbitrary successor. Obviously, this arbitrary successor can also "see" (via the accessibility relation) the successor node where C holds: this shows that not only $\Diamond p$, but $\Diamond\Diamond p$. Moreover, since the other successor has been chosen arbitrarily, we have shown that $\Box\Diamond p$ holds. In a similar way, the validity of the other axioms can be seen as well.

The reduction principles give rise to a very special *normal form* for "S5"-formulas. Let us refer to the nesting depth of $\Box$ and $\Diamond$ modalities as *modal degree.* It can be shown that every "S5" formula having a modal degree higher than one can be reduced to an equivalent "S5" formula having degree one (see [18]). The reduction principles allow us to discard all nested "$\Box$"- and "$\Diamond$"-modalities but the last one in an "S5" formula. Each "S5" formula can therefore be brought into *modal conjunctive normal form,* where each conjunct is a disjunction of the form

$$\beta \vee \Box\gamma_1 \vee \ldots \vee \Box\gamma_n \vee \Diamond\delta_1 \vee \ldots \vee \Diamond\delta_m,$$

such that all β, δ_i and γ_j are propositional formulas (see [18]). We discussed this issue briefly because we will apply a very similar idea later on in the decidability proof of $\mathcal{ALCI}_{\mathcal{RCC}3}$.

4.2 $\mathcal{ALCI}_{\mathcal{RCC}2}$ is Decidable

$\mathcal{ALCI}_{\mathcal{RCC}2}$ is decidable as well. The role box, without symmetrical entries, is the following:

$$\{DR \circ O \sqsubseteq DR \sqcup O, DR \circ DR \sqsubseteq DR \sqcup O, O \circ O \sqsubseteq DR \sqcup O\}.$$

It is obvious that every complete and $\{DR, O\}$-colored graph satisfies the role box axioms, as long as $Id(\Delta^{\mathcal{I}}) \subseteq O^{\mathcal{I}}$ holds, since we required that overlap ("O") is reflexive.

Instead of the "S5"-reduction principles, this logic includes the characteristic validities

- $\exists O.C \rightarrow \forall O.(C \sqcup \exists\{O, DR\}.C) \sqcap \forall DR.\exists\{O, DR\}.C$,
- $\exists DR.C \rightarrow \forall DR.(C \sqcup \exists\{O, DR\}.C) \sqcap \forall O.\exists\{O, DR\}.C$,
- $\forall O.C \rightarrow C$,

for all concepts C.

Regarding the complexitiy of the satisfiability problem in this logic, all we can say is that it is at least NP-hard and contained within PSPACE. Unfortunately, the standard *selection argument* which is used to show membership of "S5" in NP no longer works. Neither does encoding of the standard PSPACE-complete satisfiability problem for Quantified Boolean Formulas (QBFs). However, NP-hardness is of course an obvious lower complexity bound, since satisfiability of purely propositional formulas is already an NP-complete problem.

4.3 $\mathcal{ALCI}_{\mathcal{RCC}3}$ is Decidable

In the case of $\mathcal{ALCI}_{\mathcal{RCC}3}$ we have to distinguish between $\mathcal{ALCI}_{\mathcal{RCC}3}$ with the strong EQ-semantics and $\mathcal{ALCI}_{\mathcal{RCC}3}$ with the weak EQ-semantics: for example, $\exists EQ.(C \sqcap \exists EQ.\neg C)$ is satisfiable only under the weak EQ-semantics; under the strong semantics we would need a node i with $i = j$, $\langle i, j \rangle \in EQ^{\mathcal{I}}$ with $i \in C^{\mathcal{I}}$ and $j \in (\neg C)^{\mathcal{I}}$, which is obviously impossible. The role box gained by translating the composition table (without symmetric entries) is the following:

$$\{DR \circ ONE \sqsubseteq DR \sqcup ONE, DR \circ DR \sqsubseteq DR \sqcup ONE \sqcup EQ,\\ ONE \circ ONE \sqsubseteq DR \sqcup ONE \sqcup EQ, EQ \circ DR \sqsubseteq DR,\\ EQ \circ ONE \sqsubseteq ONE, EQ \circ EQ \sqsubseteq EQ\}.$$

4.3.1 Weak vs. Strong EQ-Semantics

It is a consequence of the composition tables that $EQ^{\mathcal{I}}$ is a *congruence relation for the binary predicates, i.e. the roles* (readers who are not familiar with the notion of a congruence relation should, for example, consult [7]). If we consider the *strong* EQ-semantics, it becomes clear that $EQ^{\mathcal{I}}$ is even a congruence relation for the *unary predicates, i.e. the concepts* as well, since $EQ^{\mathcal{I}} = Id(\Delta^{\mathcal{I}})$. Of course, a congruence relation is also an equivalence relation.

First let us show that the individuals participating in an $EQ^{\mathcal{I}}$ *equivalence class* (clique) have an identical modal point of view on the world:

Lemma 1 ((Modal) Equivalence of EQ-clique members) Let $\mathcal{I}$ be a model of an arbitrary $\mathcal{ALCI}_{\mathcal{RCC}3}$ concept C such that $\mathcal{I} \models C$, and let $i, j \in \Delta^{\mathcal{I}}$ such that $\langle i, j \rangle \in EQ^{\mathcal{I}}$. Let $\mathsf{modal\_sub}(C) =_{def} \{\exists R.D, \forall S.E \mid \exists R.D, \forall S.E \in \mathsf{sub}(C)\}$.

Then, the following holds:

- If the *weak or the strong EQ-semantics is considered,* for all $D \in \mathsf{modal\_sub}(C)$: $i \in D^{\mathcal{I}}$ iff $j \in D^{\mathcal{I}}$ – that is, i and j satisfy the same set of $\exists R.D$- and $\forall S.E$-subconcepts of C. We call i and j *modally equivalent.*
- If the *strong EQ-semantics is considered,* i and j even satisfy the same set of subconcepts: $D \in \mathsf{sub}(C)$ $i \in D^{\mathcal{I}}$ iff $j \in D^{\mathcal{I}}$ – that is, i and j are not only modally equivalent, but *equivalent.* This is, of course, obvious, since $i = j$.

Note: the stated equivalences do not only hold for concepts from $\mathsf{sub}(C)$. However, in the following it will be always sufficient to consider concepts from $\mathsf{sub}(C)$. ■

Proof 2 The latter case is trivial, and the former is easy to proof by contradiction: Suppose $\langle i,j \rangle \in EQ^{\mathcal{I}}$ and $i \in (\exists R.D)^{\mathcal{I}}$, but $j \notin (\exists R.D)^{\mathcal{I}}$. By the semantics, $i \in (\exists R.D)^{\mathcal{I}}$ means there is some $k \in D^{\mathcal{I}}$ such that $\langle i,k \rangle \in R^{\mathcal{I}}$. Since $\mathcal{I}$ is a model of the role box axioms, $\langle j,i \rangle \in EQ^{\mathcal{I}}$ and $\langle i,k \rangle \in R^{\mathcal{I}}$ implies $\langle j,k \rangle \in R^{\mathcal{I}}$. This shows that $j \in (\exists R.D)^{\mathcal{I}}$, contradicting the assumption.

Now suppose $i \in (\forall S.E)^{\mathcal{I}}$, but $j \notin (\forall S.E)^{\mathcal{I}}$. By the semantics, there must be some $k \in (\neg E)^{\mathcal{I}}$ with $\langle j,k \rangle \in S^{\mathcal{I}}$. Again, due to the role box axioms, $\langle i,j \rangle \in EQ^{\mathcal{I}}$ and $\langle j,k \rangle \in S^{\mathcal{I}}$ implies $\langle i,k \rangle \in S^{\mathcal{I}}$. However, then also $i \notin (\forall S.E)^{\mathcal{I}}$, contradicting the assumption. □

The weak EQ-semantics can be made "strong": suppose D is the concept to be tested for satisfiability under the strong EQ-semantics in $\mathcal{ALCI}_{\mathcal{RCC}3}$ with the weak EQ-semantics. To do so, just consider concept satisfiability w.r.t. the TBox that contains, for all relevant concept names $C \in \mathsf{sub}(D) \cap \mathcal{N_C}$, the axioms $C \sqsubseteq \forall EQ.C$. As already noted, TBoxes can be *internalized*: simply consider concept satisfiability of

$$D \sqcap \textstyle\bigsqcap_{C \in \mathsf{sub}(D) \cap \mathcal{N_C}} (\forall R_*.(C \rightarrow \forall EQ.C)).$$

Then, it is easy to see that all EQ-connected nodes in a model - they form an *EQ-clique* - can be collapsed into a single node, without destroying the model property. This is due to the fact that no propositional contradictions can appear when collapsing the EQ-clique members into a single node, since for all "relevant" concept names $C \in \mathsf{sub}(D) \cap \mathcal{N_C}$, $C \sqsubseteq \forall EQ.C$ holds which means that all clique members are "propositionally" equivalent (i.e. they satisfy the same set of concept names; note that the interpretations of the non-relevant concept names can simply be set to $\emptyset$).

Thus, even though it is sufficient to show decidability of $\mathcal{ALCI}_{\mathcal{RCC}3}$ w.r.t. the weak EQ-semantics since this would also show decidability of $\mathcal{ALCI}_{\mathcal{RCC}3}$ w.r.t. to the strong EQ-semantics, it is nevertheless instructive to try to give a more direct proof of this fact. We will therefore show decidability of $\mathcal{ALCI}_{\mathcal{RCC}3}$ under the strong EQ-semantics directly (this will also turn out to be much more easy), before considering the weak EQ-semantics.

4.3.2 $\mathcal{ALCI}_{\mathcal{RCC}3}$ With the Strong EQ-Semantics

If we consider $\mathcal{ALCI}_{\mathcal{RCC}3}$ with the strong EQ-semantics, then it is easy to see that we can transform each concept C into an equi-satisfiable $\mathcal{FO}_2^=$-formula; that is, first-order predicate logic with two variables and a special equality symbol "=" interpreted as identity: $(=)^{\mathcal{I}} =_{def} Id(\Delta^{\mathcal{I}})$. This is a decidable language (see, for example, [11]); therefore, $\mathcal{ALCI}_{\mathcal{RCC}3}$ with the strong EQ-semantics is decidable as well.

Given an $\mathcal{ALCI}_{\mathcal{RCC}3}$ concept C, we can exploit a variant of the so-called *standard translation* (also known from modal logics) to yield an equi-satisfiable $\mathcal{FO}_2^=$-formula $\phi_x(C)$. However, the properties of the roles must also be captured, and this is achieved with a further universally quantified closed formula θ. The translation is easy:

$\phi_x(C) =_{def} C(x)$, if C is an atomic concept
$\phi_x(\neg C) =_{def} \neg\phi_x(C)$
$\phi_x(C_1 \sqcap C_2) =_{def} \phi_x(C_1) \wedge \phi_x(C_2)$
$\phi_x(C_1 \sqcup C_2) =_{def} \phi_x(C_1) \vee \phi_x(C_2)$
$\phi_x(\exists EQ.D) =_{def} \phi_x(D)$
$\phi_x(\forall EQ.D) =_{def} \phi_x(D)$
$\phi_x(\exists R.D) =_{def} \exists y : R(x,y) \wedge \phi_y(D)$ for $R \neq EQ$
$\phi_x(\forall R.D) =_{def} \forall y : R(x,y) \rightarrow \phi_y(D)$ for $R \neq EQ$,

and θ is defined as

$$\begin{aligned}\theta =_{def}\ & \forall x,y : x = y \oplus DR(x,y) \oplus ONE(x,y) \wedge \\ & \forall x,y : DR(x,y) \leftrightarrow DR(y,x) \wedge \\ & \forall x,y : ONE(x,y) \leftrightarrow ONE(y,x).\end{aligned}$$

Note that "$\oplus$" is XOR. The definition of ϕ_y can be obtained by swapping x and y in ϕ_x. Using θ and ϕ_x, we define the $\mathcal{FO}_2^=$-translation $\gamma(C)$ of C as

$$\gamma(C) =_{def} \phi_x(C) \wedge \theta.$$ ■

Note that the strong semantics of EQ is hard-wired into the translation, by giving special translations for $\exists EQ.D$ and $\forall EQ.D$. An alternative way would be to exploit the fact that $(=)^{\mathcal{I}} =_{def} Id(\Delta^{\mathcal{I}})$ and replace the EQ-role within C by "=".

Informally, θ asserts that between any pair of *different* individuals from $\Delta^{\mathcal{I}}$, say x and y, either $DR(x,y)$ or $ONE(x,y)$ holds (and only one), that both DR and ONE are symmetric, and irreflexive.

It is exactly the irreflexiveness of DR and ONE resp. the word "different" in the above sentence for which we need "=" in the definition of θ. Obviously, $\forall x,y : EQ(x,y) \oplus DR(x,y) \oplus ONE(x,y)$ instead of $\forall x,y : x = y \oplus DR(x,y) \oplus ONE(x,y)$ doesn't enforce that EQ is interpreted as identity, such that $EQ(x,y)$ iff $x = y$.

For example,

$$\neg C \sqcap (\exists DR.\exists DR.C) \sqcap (\forall \{DR, O\}.\neg C)$$

should be unsatisfiable, but this would not be the case if we modified θ as sketched above. Moreover, replacing $\forall x,y : x = y \oplus DR(x,y) \oplus ONE(x,y)$ in θ with $\forall x,y : DR(x,y) \oplus ONE(x,y)$ doesn't work either, since, for each node, either $\langle x,x\rangle \in ONE^{\mathcal{I}}$ or $\langle x,x\rangle \in DR^{\mathcal{I}}$ would hold. Then, for example, the concept

$C \sqcap (\forall DR.\neg C) \sqcap (\forall ONE.\neg C)$ would become unsatisfiable, but the concept is obviously satisfiable under the strong EQ-semantics in $\mathcal{ALCI}_{\mathcal{RCC}3}$. We therefore need "=" for the given translation.

4.3.3 $\mathcal{ALCI}_{\mathcal{RCC}3}$ with the Weak EQ-Semantics

In order to prove decidability of $\mathcal{ALCI}_{\mathcal{RCC}3}$ *with the weak EQ-semantics*, things are no longer so easy. Considering the weak EQ-semantics, we have to ensure that
$\forall x, y, z : EQ(x, z) \leftrightarrow DR(x, y) \wedge DR(y, z) \oplus ONE(x, y) \wedge ONE(y, z) \oplus EQ(x, y) \wedge EQ(y, z)$
is satisfied, which means that EQ is really a congruence relation for the roles. It is no longer possible to simply replace EQ with "=". Even though "=" is of course also a congruence relation and therefore a valid interpretation for EQ, it is surely not the only one: Suppose that $EQ(x, y)$ holds. Under the weak EQ-semantics, x and y could very well have different propositional descriptions in $\mathcal{ALCI}_{\mathcal{RCC}3}$. As already noted, $\exists EQ.C \sqcap \exists EQ.\neg C$ is *consistent*, but translating this into $\exists x, y, z : x = y \wedge C(y) \wedge y = z \wedge \neg C(z)$ obviously yields an unsatisfiable formula.

We therefore have to *separate the modal and the propositional point of view of the EQ-connected objects.* This separation is achieved by introducing an additional binary predicate EQ'. Nested occurrences of $\exists EQ. \ldots$ and $\forall EQ. \ldots$-concepts are flattened during the translation, similar to the "S5" modal conjunctive normal form (see above) which has a modal degree of one (see above).

Definition 5 Let C be an $\mathcal{ALCI}_{\mathcal{RCC}3}$ concept in negation normal norm (NNF). Moreover, we assume that each concept C occurring within $\exists R.C$ and $\forall R.C$ is in disjunctive normal form (DNF), such that each conjunct in the disjunction of conjunctions is either an atomic concept, a negated atomic concept, or a concept of the form $\exists S.D$ or $\forall S.D$, where D is again in DNF and NFF, etc.

We then assume that there is a function α, which, applied to a disjunct D of the above DNF (note that D is itself a conjunction), returns the *modal part of* D, and that there is a corresponding function β which returns the *propositional part of* D, e.g. if $D = A_1 \sqcap (\neg A_2) \sqcap \exists R.E \sqcap \forall S.F$, then $\alpha(D) = \{\exists R.E, \forall S.F\}$ and $\beta(D) = \{A_1, (\neg A_2)\}$. We skip the easy definitions of α and β here, as well as for DNF.

The following two mutually recursive functions ϕ_x and ϕ_y do the main job (ϕ_y is obtained from ϕ_x by swapping x and y):

$$
\begin{aligned}
&\phi_x(C) =_{def} C(x), \text{ if } C \text{ is an atomic concept}\\
&\phi_x(\neg C) =_{def} \neg \phi_x(C)\\
&\phi_x(C_1 \sqcap \ldots \sqcap C_n) =_{def} \phi_x(C_1) \wedge \ldots \wedge \phi_x(C_n)\\
&\phi_x(C_1 \sqcup \ldots \sqcup C_n) =_{def} \phi_x(C_1) \vee \ldots \vee \phi_x(C_n)\\
&\phi_x(\exists EQ.C) =_{def} \ (\textstyle\bigwedge_{mp \in \alpha(C)} \phi_x(mp) \wedge\\
&\qquad (\exists y : EQ'(x, y) \wedge EQN(y) \wedge \textstyle\bigwedge_{bp \in \beta(C)} \phi_y(bp))\\
&\qquad \vee \phi_x(C)), \text{ if } C \text{ is not a disjunction}
\end{aligned}
$$

$\phi_x(\exists EQ.(C_1 \sqcup \ldots \sqcup C_n)) =_{def} \phi_x(\exists EQ.C_1) \vee \ldots \vee \phi_x(\exists EQ.C_n)$
$\phi_x(\exists R.C) =_{def} (\exists y : R(x,y) \wedge \phi_y(\exists EQ.C))$, if $R \neq EQ$
$\phi_x(\forall EQ.C) =_{def} \; (\bigwedge_{mp \in \alpha(C)} \phi_x(mp) \wedge$
$(\forall y : EQ'(x,y) \wedge EQN(y) \rightarrow \bigwedge_{bp \in \beta(C)} \phi_y(bp))$
$\wedge \phi_x(C))$, if C is not a disjunction
$\phi_x(\forall EQ.(C_1 \sqcup \ldots \sqcup C_n)) =_{def} \neg(\phi_x(\exists EQ.\mathsf{DNF}(\mathsf{NNF}((\neg C_1 \sqcap \ldots \sqcap \neg C_n)))))$
$\phi_x(\forall R.C) =_{def} (\forall y : R(x,y) \rightarrow \phi_y(\forall EQ.C))$, if $R \neq EQ$

The formula θ is now defined as follows (please note that we have fixed some errors that were present in θ as given in [27]):

$$\begin{aligned}\theta =_{def} \; & \forall x,y : x \neq y \wedge \neg EQN(x) \wedge \neg EQN(y) \leftrightarrow (DR(x,y) \vee ONE(x,y)) \wedge \\ & \forall x,y : \neg(DR(x,y) \wedge ONE(x,y)) \wedge \\ & \forall x,y : DR(x,y) \leftrightarrow DR(y,x) \wedge \\ & \forall x,y : ONE(x,y) \leftrightarrow ONE(y,x) \wedge \\ & \forall x,y : EQ'(x,y) \rightarrow \neg EQN(x) \wedge EQN(y).\end{aligned}$$

Informally, θ asserts that between any pair of individuals x and y, none of which is marked with EQN, either $DR(x,y)$ or $ONE(x,y)$ holds (but not both), that both DR and ONE are symmetric, irreflexive, and that there is an other category of individuals with is marked with EQN, which are connected to non-EQN-objects via the EQ'-role (predicate).

Using θ and ϕ_x, we define the $\mathcal{FO}_2^=$-translation $\gamma(C)$ of C as

$$\gamma(C) =_{def} \phi_x(C) \wedge \theta.$$ ■

Theorem 1 C is satisfiable in $\mathcal{ALCI}_{\mathcal{RCC}3}$ under the weak EQ-semantics iff the $\mathcal{FO}_2^=$-formula $\gamma(C)$ is. ■

Proof 3 Given a model $\mathcal{I}$ for the concept term C, we show how to construct a model $\mathcal{I}'$ such that $\mathcal{I}' \models \gamma(C) = \phi_x(C) \wedge \theta$. An illustration of the model re-construction process is given in Figure 7.

Since $\mathcal{I}'$ is a model of a first-order formula with a free variable x, we also need to supply a valuation for x using the function χ (mapping variables to individuals in $\Delta^{\mathcal{I}'}$).

On the other hand, if $\mathcal{I}' \models \gamma(C) = \phi_x(C) \wedge \theta$ we show how to construct a model for the original concept term C such that $\mathcal{I} \models C$.

So let $\mathcal{I} \models C$, with $\mathcal{I} = (\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}})$. Please note that $EQ^{\mathcal{I}}$ is an equivalence relation (even a congruence relation). Denote the set of equivalence classes of $EQ^{\mathcal{I}}$ in $\Delta^{\mathcal{I}}$ with $\Delta^{\mathcal{I}}_{/EQ}$, and assume that, for each $[a] \in \Delta^{\mathcal{I}}_{/EQ}$, there is a *fixed ordering* of the elements in this equivalence class, e.g. for $i \in 1 \ldots \#[a]$ ($\#[a]$ denotes the cardinality of $[a]$), $[a] \in \Delta^{\mathcal{I}}_{/EQ}$, $[a]_i$ denotes the (unique!) ith element in the equivalence class $[a]$.

Then, $\mathcal{I}' = (\Delta^{\mathcal{I}'}, \cdot^{\mathcal{I}'}, \chi)$ is defined as follows:

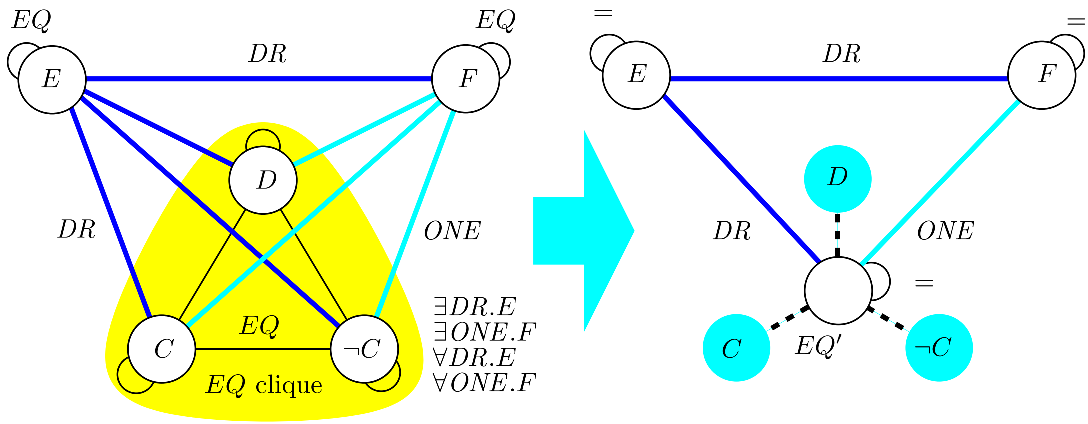


Figure 7: On the left, a model of an $\mathcal{ALCI}_{\mathcal{RCC}3}$ concept term under the weak EQ-semantics. On the right hand side a model of its $\mathcal{FO}_2^=$-translation. Illustration of an EQ-clique: Clique-members have an equivalent modal point of view. One the right, the EQ-clique has been collapsed into a single node. Note that collapsing an EQ-clique can never violate the role disjointness requirement, since EQ is a congruence relation for the roles. On the right-hand side, also note the additional (shaded) EQN-nodes, representing the original EQ-clique members.

- $\Delta^{\mathcal{I}'} =_{def} \Delta^{\mathcal{I}}_{/EQ} \cup \{\, [a]_i \mid i \in 1 \ldots \#[a], [a] \in \Delta^{\mathcal{I}}_{/EQ} \,\}$.
  This means there is an individual $[a]$ in $\Delta^{\mathcal{I}'}$ for each EQ-equivalence class $[a]$, as well as one individual $[a]_i$ for each member $a_i \in [a]$ in this class. Also note that $[a]$ denotes two different things: first, the whole equivalence class (i.e., a set of individuals $\{[a]_1, \ldots, [a]_{(\#[a])}\}$), as well as an individual $[a]$ in $\Delta^{\mathcal{I}'}$ *representing* the class $[a]$. It should be clear from the context which one is meant;
- $DR^{\mathcal{I}'} =_{def} \{\, <[a],[b]> \mid <a,b> \in DR^{\mathcal{I}} \,\}$;
- $ONE^{\mathcal{I}'} =_{def} \{\, <[a],[b]> \mid <a,b> \in ONE^{\mathcal{I}} \,\}$;
- $EQ'^{\mathcal{I}'} =_{def} \{\, <[a],[a]_i> \mid i \in 1 \ldots \#[a], [a] \in \Delta^{\mathcal{I}}_{/EQ} \,\}$;
- $(=)^{\mathcal{I}'} =_{def} \{\, <x,x> \mid x \in \Delta^{\mathcal{I}'} \,\}$;
- for all concept names $D \in \mathcal{N_C}$, $[a]_i \in D^{\mathcal{I}'}$ iff $a \in D^{\mathcal{I}}$ and a is the ith element in the fixed ordering in its equivalence class;
- $EQN^{\mathcal{I}'} =_{def} \{\, [a]_i \mid i \in 1 \ldots \#[a], [a] \in \Delta^{\mathcal{I}}_{/EQ} \,\}$.

We first show that, given an arbitrary model $\mathcal{I}$ that satisfies the role axioms of $\mathcal{ALCI}_{\mathcal{RCC}3}$, the model $\mathcal{I}'$ will also satisfy θ. We do this by showing that every conjunct of θ is satisfied.

- $\forall x, y : x \neq y \wedge \neg EQN(x) \wedge \neg EQN(y) \leftrightarrow (DR(x,y) \vee ONE(x,y))$:

"$\leftarrow$" If $(DR(x,y) \vee ONE(x,y))$ for two nodes $x, y \in \Delta^{\mathcal{I}'}$, then, by definition of $DR^{\mathcal{I}'}$ and $ONE^{\mathcal{I}'}$, $x \neq y$, since $DR^{\mathcal{I}}$ and $ONE^{\mathcal{I}}$ are irreflexive. Since $x = [a]$ and $y = [b]$ for some $a, b \in \Delta^{\mathcal{I}}$ and $[a] \neq [a]_i$ for all i, also $\neg EQN(x) \wedge \neg EQN(y)$, since $EQN(x)$ iff $x = [a]_i$, for some $[a]$ and i.

"$\rightarrow$" If $x \neq y \wedge \neg EQN(x) \wedge \neg EQN(y)$ for two nodes $x, y \in \Delta^{\mathcal{I}'}$, then, $x = [a]$ and $x = [b]$, with $[a] \neq [b]$. Therefore, $<a, b> \notin EQ^{\mathcal{I}}$. Because $\mathcal{I}$ satisfies the role box axioms and is a complete graph, either $<a, b> \in DR^{\mathcal{I}}$ or $<a, b> \in ONE^{\mathcal{I}}$. By definition of $DR^{\mathcal{I}'}$ and $ONE^{\mathcal{I}'}$, also $(DR(x,y) \vee ONE(x,y))$.

- $\forall x, y : \neg(DR(x,y) \wedge ONE(x,y))$: due to the fact that $DR^{\mathcal{I}} \cap ONE^{\mathcal{I}} = \emptyset$, and that $EQ^{\mathcal{I}}$ is a congruence relation for the roles (this implies that the replacement of the nodes in $\mathcal{I}$ by their equivalence classes in $\mathcal{I}'$ never yields non-empty role intersections).

- $\forall x, y : DR(x,y) \leftrightarrow DR(y,x)$: an obvious consequence of the semantics of $\mathcal{ALCI}_{\mathcal{RCC}3}$ and of the fact that $EQ^{\mathcal{I}}$ is a congruence relation for the roles.

- $\forall x, y : ONE(x,y) \leftrightarrow ONE(y,x)$: see above.

- $\forall x, y : EQ'(x,y) \rightarrow \neg EQN(x) \wedge EQN(y)$: is easy to satisfy by *minimizing* $EQ'^{\mathcal{I}'}$.

Using induction on C, we show that $\mathcal{I}' \models \gamma(C) = \phi_x(C) \wedge \theta$. Since we have just shown that $\mathcal{I}' \models \theta$, by definition of $\mathcal{I}'$, we subsequently only show that $\mathcal{I}' \models \phi_x(C)$:

- If $C \in \mathcal{N_C}$ (i.e., C is a concept name) and $\mathcal{I} \models C$, then there is some $i \in \Delta^{\mathcal{I}}$ with $i \in C^{\mathcal{I}}$. Since $i \in [i]$, there is some $[i]_i \in \Delta^{\mathcal{I}'}$. W.l.o.g. we assume that i is the ith element in $[i]$. Since $i \in C^{\mathcal{I}}$ we also have $[i]_i \in C^{\mathcal{I}'}$, by definition. By assigning $\chi(x) = [i]_i$ we see that $\mathcal{I}' \models \phi_x(C) = C(x)$.

- If $C = \neg D$, then $D \in \mathcal{N_C}$, i.e. D is an atomic concept. See above. Note that $[i]_i \in D^{\mathcal{I}'}$ *if and only if* $i \in D^{\mathcal{I}}$ and i is the ith element in the fixed ordering in its equivalence class.

- If $C = C_1 \sqcap C_2$, $\mathcal{I} \models C_1 \sqcap C_2$, then, by the semantics of "$\sqcap$", there is some $i \in C_1^{\mathcal{I}}$, $i \in C_2^{\mathcal{I}}$. By the induction hypothesis, there is some $[i]_i \in \Delta^{\mathcal{I}'}$ with $\chi(x) = [i]_i$ such that $\mathcal{I}' \models \phi_x(C_1)$ and $\mathcal{I}' \models \phi_x(C_2)$. But then, due to the semantics of "$\wedge$", also $\mathcal{I}' \models \phi_x(C_1) \wedge \phi_x(C_2)$, thus $\mathcal{I}' \models \gamma(C)$, since both $\phi_x(C_1)$ and $\phi_x(C_2)$ have the same free variable x.

- If $C = C_1 \sqcup C_2$, $\mathcal{I} \models C_1 \sqcup C_2$, then, by the semantics of "$\sqcup$", there is some $i \in C_1^{\mathcal{I}}$ or some $i \in C_2^{\mathcal{I}}$. By the induction hypothesis, there is some $[i]_i \in \Delta^{\mathcal{I}'}$ with $\chi(x) = [a]_i$ such that $\mathcal{I}' \models \phi_x(C_1)$ or $\mathcal{I}' \models \phi_x(C_2)$. But then, due to the semantics of "$\vee$", also $\mathcal{I}' \models \phi_x(C_1) \vee \phi_x(C_2)$, thus $\mathcal{I}' \models \phi_x(C)$, since both $\phi_x(C_1)$ and $\phi_x(C_2)$ have the same free variable x.

- If $C = (\exists EQ.D)$ and D is not a disjunction, then its translation is

  $$\begin{array}{ll}\phi_x(\exists EQ.D) =_{def} & (\bigwedge_{mp\in\alpha(D)} \phi_x(mp) \wedge \\ & (\exists y : EQ'(x,y) \wedge EQN(y) \wedge \bigwedge_{bp\in\beta(D)} \phi_y(bp)) \\ & \vee \phi_x(D)).\end{array}$$

  Since $\mathcal{I} \models \exists EQ.D$ there are $i \in (\exists EQ.D)^{\mathcal{I}}$ and $j \in D^{\mathcal{I}}$ with $<i,j> \in EQ^{\mathcal{I}}$. Note that $[i] = [j]$; i.e. i and j are members in the same EQ-equivalence class; however, i and j must not be identically. Due to the induction hypothesis, there is some $[j]_j$ such that $\chi(x) = [j]_j$ and $\mathcal{I}' \models \phi_x(D)$.

  Now, there are two possibilities:

  – If $i = j$ due to the reflexiveness of $EQ^{\mathcal{I}}$, thus $[i]_i = [j]_j$, we have $\mathcal{I}' \models \phi_x(C)$, since $\mathcal{I}' \models \phi_x(D)$ with $\chi(x) = [i]_i$ and $\phi_x(C) = \ldots \vee \phi_x(D)$.

  – If $i \neq j$, then by construction, $j \in [i]$ since $[i] = [j]$ due to $<i,j> \in EQ^{\mathcal{I}}$. By construction of $\mathcal{I}'$ we know that $<[i],[i]_j> \in EQ'^{\mathcal{I}'}$, thus $<[i],[j]_j> \in EQ'^{\mathcal{I}'}$, as well as $[j]_j \in EQN^{\mathcal{I}'}$.

    Consider now $\sqcap_{mp\in\alpha(D)} mp$. As defined above, $\alpha(D)$ returns the *modal part* of D; i.e. the set of some- and forall-concepts appearing in D. Note that this set is semantically well-defined: by definition, D does not contain any disjunctions (the "next" disjunction might only appear within "E" in a $\exists R.E$- or $\forall R.E$-conjunct of D). Since $\alpha(D)$ is a conjunction of conjuncts appearing in D (i.e., a sub-conjunction of D), also $j \in (\sqcap_{mp\in\alpha(D)})^{\mathcal{I}}$. But then, due to Lemma 1, also $i \in (\sqcap_{mp\in\alpha(D)})^{\mathcal{I}}$ – recall that $EQ^{\mathcal{I}}$ is not only an equivalence, but a *congruence relation* for the roles (binary predicates). This shows that we can assign $\chi(x) = [i]$, and $\mathcal{I}' \models \bigwedge_{mp\in\alpha(D)} \phi_x(mp)$.

    Consider now $\sqcap_{bp\in\beta(D)} bp$. As defined above, $\beta(D)$ returns the *boolean part* of D. Obviously, $D = \bigsqcap_{x\in\alpha(D)\cup\beta(D)} x$ (up to reordering of conjuncts). If we set $\chi(y) = [j]_j$, then $\mathcal{I}' \models \phi_y(D)$ and therefore $\mathcal{I}' \models (\exists y : EQ'(x,y) \wedge EQN(y) \wedge \bigwedge_{bp\in\beta(D)} \phi_y(bp))$ if we assign $\chi(x) = [i]$. Note that by definition of $\mathcal{I}'$, $<[i],[i]_j> \in EQ'^{\mathcal{I}'}$, as well as $[i]_j \in EQN^{\mathcal{I}'}$.

- If $C = (\exists EQ.(C_1 \sqcup \ldots \sqcup C_n))$, then its translation is given by $\phi_x(\exists EQ.(C_1 \sqcup \ldots \sqcup C_n)) =_{def} \phi_x(\exists EQ.C_1) \vee \ldots \vee \phi_x(\exists EQ.C_n)$.

  Immediate by the semantics. Note that each C_i is itself a conjunction (possibly of length one) - $\phi_x(\exists EQ.C_i)$ is therefore covered by the previous case.

- If $C = \exists R.D$, $R \in \{DR, ONE\}$, then its translation is given by $\phi_x(\exists R.D) =_{def} (\exists y : R(x,y) \wedge \phi_y(\exists EQ.D))$. If $i \in (\exists R.D)^{\mathcal{I}}$, then there is some $j \in D^{\mathcal{I}}$ such that $<i,j> \in R^{\mathcal{I}}$. By the induction hypothesis, there is some $\mathcal{I}'$ with $\chi(x) = [j]_j$ $\mathcal{I}' \models \phi_x(D)$. Then also $\mathcal{I}' \models \phi_y(D)$ with $\chi(y) = [j]_j$. Since $[j]_j \in [j]$, $<[j],[j]_j> \in EQ'^{\mathcal{I}'}$ we can show that $\mathcal{I}' \models \phi_y(\exists EQ.D)$ for $\chi(y) = [j]$. Since $<[i],[j]> \in R^{\mathcal{I}'}$ we can assign $\chi(x) = [i]$ and get $\mathcal{I}' \models (\exists y : R(x,y) \wedge \phi_y(\exists EQ.D))$.

- If $C = (\forall EQ.D)$ and D is not a disjunction, then its translation is

$$\begin{array}{ll}\phi_x(\forall EQ.D) =_{def} & (\bigwedge_{mp\in\alpha(D)} \phi_x(mp) \wedge \\ & (\forall y : EQ'(x,y) \wedge EQN(y) \rightarrow \bigwedge_{bp\in\beta(D)} \phi_y(bp)) \\ & \wedge \phi_x(D))\end{array}$$

Let $i \in (\forall EQ.D)^{\mathcal{I}}$ and $j \in D^{\mathcal{I}}$ such that $\langle i,j\rangle \in EQ^{\mathcal{I}}$. By definition of $\mathcal{I}'$, $[i] = [j]$, i.e. i and j are members in the same EQ-equivalence class. Due to the induction hypothesis, $\mathcal{I}' \models \phi_x(D)$ with $\chi(x) = [i]_j$, as well as for $\chi(x) = [i]_i$ and $\chi(x) = [i]$, since $[i] \subseteq D^{\mathcal{I}}$ (i.e., all members of $[i]$ are instances of D). Now, $(\bigsqcap_{mp\in\alpha(D)} mp)$ is a sub-conjunction of D, thus we have $[i] \subseteq (\bigsqcap_{mp\in\alpha(D)} bp) \sqcap (\bigsqcap_{bp\in\beta(D)} bp)^{\mathcal{I}}$. This shows that $\mathcal{I}' \models (\bigwedge_{mp\in\alpha(D)} \phi_x(mp))$ for $\chi(x) = [i]$ as well as for $\chi(x) = [i]_i$, $\chi(x) = [i]_j$, etc. Therefore, $\mathcal{I}' \models (\bigwedge_{bp\in\alpha(D)} \phi_x(bp))$. By definition of $\mathcal{I}'$, $\langle [i],[i]_j\rangle \in EQ'^{\mathcal{I}'}$ and $[i]_j \in EQN^{\mathcal{I}'}$. But then, $\mathcal{I}' \models (\forall y : EQ'(x,y) \wedge EQN(y) \rightarrow \bigwedge_{bp\in\beta(D)} \phi_y(bp))$ for $\chi(x) = [i]$, thus $\mathcal{I}' \models \phi_x(C)$.

- If $C = \forall EQ.(C_1 \sqcup \ldots \sqcup C_n)$, then its translation is given by $\phi_x(\forall EQ.(C_1 \sqcup \ldots \sqcup C_n)) =_{def} \neg(\phi_x(\exists EQ.\mathsf{DNF}(\mathsf{NNF}((\neg C_1 \sqcap \ldots \sqcap \neg C_n)))))$. This holds, since $\forall R.D \equiv \neg\exists R.\neg D$ is an immediate consequence of the semantics, and $\neg(C_1 \sqcup C_2) \equiv (\neg C_1 \sqcap \neg C_2)$. Obviously, NNF and DNF preserve satisfiability. Please note that $\mathsf{DNF}(\mathsf{NNF}(\neg C_1 \sqcap \ldots \sqcap \neg C_n))$ returns a concept which has again the required normal form properties, such that $\phi_x(\exists EQ.\ldots)$ can be exploited for the translation.

- If is $C = \forall R.D$, $R \in \{DR, ONE\}$, then its translation is given by $\phi_x(\forall R.D) =_{def} (\forall y : R(x,y) \rightarrow \phi_y(\forall EQ.D))$. If $i \in (\forall R.D)^{\mathcal{I}}$ and $\langle i,j\rangle \in R^{\mathcal{I}}$, then, by the semantics, $j \in D^{\mathcal{I}}$. By the induction hypothesis there is some $\mathcal{I}'$ with $\chi(x) = [j]_j$, $\mathcal{I}' \models \phi_x(D)$ (if we assume that j is the jth node in its equivalence class). Since for all nodes $j,k \in [j]$ $\langle j,k\rangle \in EQ^{\mathcal{I}}$ and $R \circ EQ \sqsubseteq R$ we know that $\langle i,k\rangle \in R^{\mathcal{I}}$. But then, $[j] \subseteq D^{\mathcal{I}}$. This shows that $j \in (\forall EQ.D)^{\mathcal{I}}$, $[j]_j \in (\forall EQ.D)^{\mathcal{I}}$, as well as $[j] \subseteq (\forall EQ.D)^{\mathcal{I}}$. By the semantics and the induction hypothesis this shows that $\mathcal{I}' \models \phi_y(\forall EQ.D)$ with $\chi(y) = [j]_j$. Thus we have shown $\mathcal{I}' \models (\forall y : R(x,y) \rightarrow \phi_y(\forall EQ.D))$ with $\chi(x) = [i]$.

The other direction in the proof is to show by induction on C that any model for $\mathcal{I} \models \gamma(C) = \phi_x(C) \wedge \theta$ (with $\chi(x) = i$) can be transformed into a model $\mathcal{I}'$ of C. The proof is very similar; just the line of argumentation needs to be "reversed". We just show how to construct $\mathcal{I}'$ from $\mathcal{I}$. Let $\mathcal{I} = (\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}}, \chi)$ be a model of $\gamma(C)$ with a *minimal* interpretation of $EQN^{\mathcal{I}}$ and $EQ'^{\mathcal{I}}$. Then $\mathcal{I}' = (\Delta^{\mathcal{I}'}, \cdot^{\mathcal{I}'})$ is defined as follows:

- $\Delta^{\mathcal{I}'} =_{def} \Delta^{\mathcal{I}}$;
- $EQ^{\mathcal{I}'} =_{def} (EQ'^{\mathcal{I}})^{\otimes}$; where "$\otimes$" denotes the reflexive, transitive and *symmetric* closure of a binary relation;
- $R^{\mathcal{I}'} =_{def} \{ \langle i,k\rangle \mid i,j,k \in \Delta^{\mathcal{I}'}, \langle j,k\rangle \in EQ^{\mathcal{I}'}, \langle i,j\rangle \in R^{\mathcal{I}} \}$ for $R \in \{DR, ONE\}$;
- for all concept names $D \in \mathcal{N}_{\mathcal{C}}$, $D^{\mathcal{I}'} =_{def} D^{\mathcal{I}}$.

Is is readily checked that $\mathcal{I}'$ validates the role box axioms and semantic constraints on the interpretations of the roles (e.g., that $EQ^{\mathcal{I}'}$ is a congruence relation for $\{DR, ONE\}$), etc. We omit the inductive proof that $\mathcal{I}' \models C$. $\square$

4.4 Beyond $\mathcal{ALCI}_{\mathcal{RCC}3}$: $\mathcal{ALCI}_{\mathcal{RCC}5}$ and $\mathcal{ALCI}_{\mathcal{RCC}8}$

Considering $\mathcal{ALCI}_{\mathcal{RCC}5}$ and $\mathcal{ALCI}_{\mathcal{RCC}8}$, we can observe that neither has the finite model property, unlike $\mathcal{ALCI}_{\mathcal{RCC}3}$ and its sub-languages (it is well known that $\mathcal{FO}_2^=$ has the finite model property). We've already noted that, for example, $((\exists PP.\top) \sqcap (\forall PP.\exists PP.\top))$ has no finite models.

There is some indication that $\mathcal{ALCI}_{\mathcal{RCC}5}$ could possibly be computationally easier to handle than $\mathcal{ALCI}_{\mathcal{RCC}8}$, since the latter seems to have more expressive power. More specifically, unlike $\mathcal{ALCI}_{\mathcal{RCC}5}$, $\mathcal{ALCI}_{\mathcal{RCC}8}$ somehow allows the distinction of a role and its *transitive orbit*. A transitive orbit is a role whose interpretation contains *at least* the transitive closure of the interpretation of the generating role, i.e. if $\oplus(R)$ returns the role that has to be interpreted as a transitive orbit of the role R, then the semantics requires that $(R^{\mathcal{I}})^+ \subseteq (\oplus(R))^{\mathcal{I}}$, see [23]. Once a role and its transitive orbit can be distinguished it becomes possible to encode models with paths of exponential length, see below.

4.4.1 PSPACE-hardness of $\mathcal{ALCI}_{\mathcal{RCC}5}$

Currently we can only show PSPACE-hardness, which follows from *Ladner's Theorem.* It is possible to reduce the validity problem for QBFs (Quantified Boolean Formulas) to the satisfiability problem in $\mathcal{ALCI}_{\mathcal{RCC}5}$. To do so, we only have to show how to enforce the existence of *tree models*. We do so by *forbidding* the PO relation to hold between nodes in the model. Despite some technical tricks, the following formulas are nearly identical to the ones in [7, pp. 383]. In order to emphasize the close relatedness of DLs and modal logics, we will use modal logics syntax this time.

Let $B_i =_{def} q_i \rightarrow (\Diamond_{PPI}(q_{i+1} \wedge p_{i+1}) \wedge \Diamond_{PPI}(q_{i+1} \wedge \neg p_{i+1}))$, and $S(p_i, \neg p_i) =_{def} (p_i \rightarrow \Box_{PPI}\ p_i) \wedge (\neg p_i \rightarrow \Box_{PPI}\ \neg p_i)$, and consider the following conjunction:

- Root node: q_0
- Forbid PO: $\Box_{R_*}\Box_{PO}\bot$ (note: same as $\forall R_*.\forall PO.\bot$)
- Disjointness of q_i's: $\Box_{R_*}(q_i \rightarrow \bigwedge_{i \neq j, 0 \leq i \leq m} \neg q_j)$
- Ensure DR between siblings: $\Box_{R_*}(q_i \rightarrow \Box_{PPI}\neg q_i) \quad 0 \leq i \leq m$
- Binary branching: $B_0 \wedge \Box_{PPI} B_1 \wedge \Box^2_{PPI} B_2 \wedge \cdots \wedge \Box^{m-1}_{PPI} B_{m-1}$
- Consistent propagation of "bits" along the tree:

$$\begin{array}{lllll} \Box_{PPI} S(p_1, \neg p_1) & \wedge \; \Box^2_{PPI} S(p_1, \neg p_1) & \wedge \cdots \wedge & \Box^{m-1}_{PPI} S(p_1, \neg p_1) & \wedge \\ & \wedge \; \Box^2_{PPI} S(p_2, \neg p_2) & \wedge \cdots \wedge & \Box^{m-1}_{PPI} S(p_2, \neg p_2) & \wedge \\ & & & \vdots & \\ & & & \Box^{m-1}_{PPI} S(p_{m-1}, \neg p_{m-1}) & \end{array}$$

If we consider the strong EQ semantics, the enforced models are binary transitive trees of height m, with $DR^{\mathcal{I}} = \Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}} \setminus (PP^{\mathcal{I}} \cup PPI^{\mathcal{I}} \cup Id(\Delta^{\mathcal{I}}))$, $PPI^{\mathcal{I}} = (PPI^{\mathcal{I}})^+$ and $PP^{\mathcal{I}} = (PPI^{\mathcal{I}})^{-1}$. As already noted, the weak EQ semantics can be made strong, if we add $\Box_{R*}(p \rightarrow \Box_{EQ}\ p)$ for all proposition letters, and collapse the EQ cliques in the model. This means, we have a super-logic of $K4$ with a "difference modality" DR, which shows PSPACE-hardness as a lower bound; the upper bound is unknown to us. However if we consider the "PO-free fragment" of $\mathcal{ALCI}_{\mathcal{RCC}5}$ it becomes reasonable to conjecture that this logic should be in PSPACE and be therefore PSPACE-complete.

It is interesting to note that we can also use $\mathcal{ALCI}_{\mathcal{RCC}5}$ as a *propositional linear tense logic* (Priorean tense logic, PTL) – which is basic modal logic with the P and F modalities resp. their duals H and G) – interpreted in *strict total orders*. Intuitively, "$F\phi$" means "somewhere in the $f$uture, ϕ will hold", and "$G\phi$" means "it is always $g$oing to be the case that ϕ holds", and accordingly for the past. Obviously, $\Diamond_{PPI}$ can be used instead of the F operator, and $\Diamond_{PP}$ as the P operator. It is also easy to enforce that $PP^{\mathcal{I}}$ and $PPI^{\mathcal{I}}$ are linear strict orders: just add the conjunct $\Box_{R \in \{DR,EC,PO\}} \bot$ *globally* to the original formula; e.g. we translate $living\_being \rightarrow F(dead \wedge G\ dead)$ into $(living\_being \rightarrow \Diamond_{PPI}(dead \wedge \Box_{PPI}\ dead)) \wedge \Box_{R*}\ \Box_{R \in \{DR,EC,PO\}} \bot$. Additional conjuncts might be added for denseness, left- and right-(un)boundedness, etc.

4.4.2 EXPTIME-hardness of $\mathcal{ALCI}_{\mathcal{RCC}8}$

Once a role can be distinguished from its transitive orbit (or even its exact transitive closure), which is the case for all super-languages of $\mathcal{ALC}_{\mathcal{R}\oplus}$ (see [23]), e.g. languages like $\mathcal{ALCH}_{\mathcal{R}^+}$, $\mathcal{SHIQ}$ (see [17]), PDL (Propositional Dynamic Logic) etc., it is easy to construct concepts enforcing models having *paths of exponential length in the size of the input concept.* Note that this is in contrast to the previous QBF reduction - even though the enforced tree models were also exponential, its *tree paths* were still polynomial (linear) in the size of the input formula.

We first show how to distinguish a role from its transitive orbit in $\mathcal{ALCI}_{\mathcal{RCC}8}$. The following concept enforces an infinite chain of *even-odd-*…-marked individuals, see Figure 8 and the spatial illustration given in Figure 9. Each node can distinguish its direct $TPPI$-successor from all its indirect $NTPPI$-successors. Laxly speaking, we can consider $NTPPI$ somehow as the transitive orbit of $TPPI$; more specifically, we have $((TPPI^{\mathcal{I}})^+ - TPPI^{\mathcal{I}}) \subseteq NTPPI^{\mathcal{I}}$. An "exact" transitive orbit would require $(TPPI^{\mathcal{I}})^+ \subseteq NTPPI^{\mathcal{I}}$, but the construction is fine for what follows.

The construction is the following:

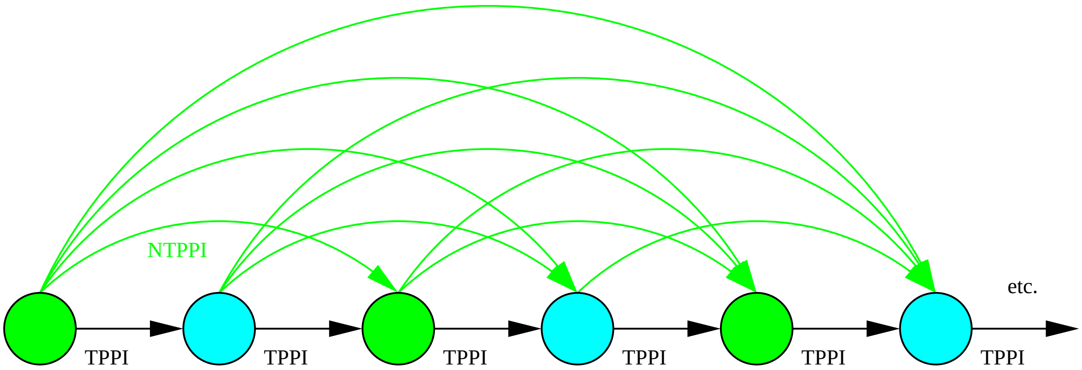


Figure 8: Illustration of a (part of an infinite) model of *infinite_even_odd_chain*

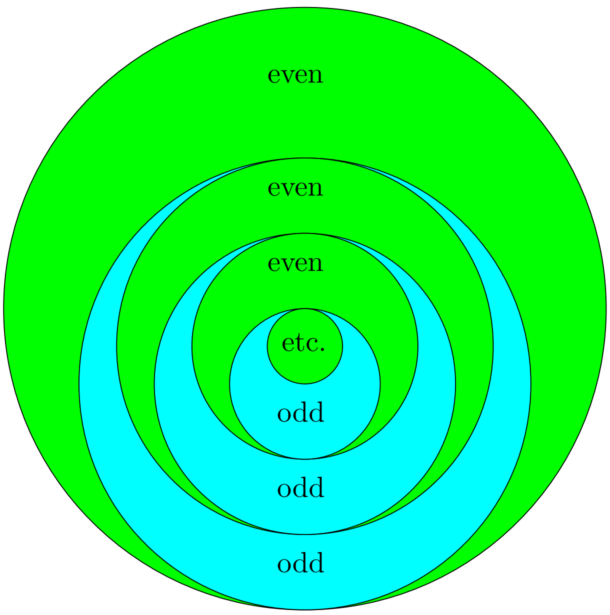


Figure 9: Spatial illustration of *infinite_even_odd_chain*

$$
\begin{array}{rl}
infinite\_even\_odd\_chain =_{def} & even \sqcap \\
& even \rightarrow \forall TPPI.odd \sqcap \\
& \forall R_*.\ ((even \leftrightarrow \neg odd) \sqcap \\
& \qquad (even \rightarrow \forall TPPI.odd) \sqcap \\
& \qquad (odd \rightarrow \forall TPPI.even)) \sqcap \\
& \exists TPPI.\exists TPPI.\top \sqcap \\
& \forall NTPPI.\exists TPPI.\top
\end{array}
$$

Due to the RCC8 composition table, we have

$$
\begin{array}{lcllcl}
TPPI & \circ & TPPI & \sqsubseteq & TPPI \sqcup NTPPI \\
TPPI & \circ & NTPPI & \sqsubseteq & NTPPI \\
NTPPI & \circ & TPPI & \sqsubseteq & NTPPI \\
NTPPI & \circ & NTPPI & \sqsubseteq & NTPPI.
\end{array}
$$

Now that we know how to separate a role from its (pseudo-) transitive orbit we can continue by showing how to encode the classical *binary n-bit counter*: if the binary counter has n bits, then each of its models must have a path of a length at least 2^n, representing all numbers from 0 to $2^n - 1$, one individual along the path for each number (see also [7] for an encoding of this concept in PDL). Please note that *transitively closed roles* are *not* sufficient for an encoding of the counter. The counter concept in $\mathcal{ALCI}_{\mathcal{RCC}8}$ looks like that:

$$
\begin{array}{rl}
counter =_{def} & even \sqcap \\
& even \rightarrow \forall TPPI.odd \sqcap \\
& \forall R_*.((even \leftrightarrow \neg odd) \sqcap (even \rightarrow \forall TPPI.odd) \sqcap (odd \rightarrow \forall TPPI.even)) \sqcap \\
& \forall NTPPI.(\neg(bit_0 \sqcap \ldots \sqcap bit_{n-1}) \rightarrow \exists TPPI.\top) \sqcap \\
& \exists TPPI.\exists TPPI.(\neg bit_0 \sqcap \neg bit_1 \sqcap \ldots \sqcap \neg bit_{n-1}) \sqcap \\
& \forall NTPPI.(toggle\_bit_0 \sqcap toggle\_bit_1 \sqcap \ldots \sqcap toggle\_bit_{n-1})
\end{array}
$$

$$
toggle\_bit_0 =_{def} \ (bit_0 \sqcap \forall TPPI.\neg bit_0) \sqcup (\neg bit_0 \sqcap \forall TPPI.bit_0)
$$

$$
\begin{array}{rl}
toggle\_bit_i =_{def} & ((\sqcap_{0 \le j < i} bit_j) \sqcap ((bit_i \sqcap \forall TPPI.\neg bit_i) \sqcup (\neg bit_i \sqcap \forall TPPI.bit_i))) \sqcup \\
& (\neg(\sqcap_{0 \le j < i} bit_j) \sqcap ((bit_i \sqcap \forall TPPI.bit_i) \sqcup (\neg bit_i \sqcap \forall TPPI.\neg bit_i)))
\end{array}
$$

Informally, the first three conjuncts in *counter* enforce an alternating toggling of $even - odd$-marked *TPPI*-successor nodes. As seen before this gives us $((TPPI^{\mathcal{I}})^+ - TPPI^{\mathcal{I}}) \subseteq NTPPI^{\mathcal{I}}$. The fourth conjunct says that unless the "highest number" of the counter $2^n - 1$ has been reached (which means that all bits are "turned on"), yet an other *TPPI*-successor is still needed. The fifth conjunct ensures the existence of a node which represents the number zero (all bits are set to zero). The sixth conjunct is used to propagate the $toggle\_bit_i$-concepts to each node along the path. The $toggle\_bit_i$-concepts do exactly what their naming suggests - they enforce the correct incrementation (bit toggling) along the *TPPI*-path. For example, $toggle\_bit_1$ turns bit_1 of the next *TPPI*-successor on iff currently bit_0 is on and bit_1 is off (note that bit_0 is turned off at the *TPPI*-successor by $toggle\_bit_0$). Bit_1 is reset to zero again at the *TPPI*-successor iff both bit_0 and bit_1 are currently on. Otherwise, the state of bit_1 is left unchanged.

Obviously, each model must have at least $2 + 2^n$ nodes ("+2" because the zero starts at the second *TPPI*-successor, $\exists TPPI.\exists TPPI.(\neg bit_0 \sqcap \neg bit_1 \sqcap \ldots \sqcap \neg bit_{n-1})$). There does not seem to be a way to achieve a similar effect in $\mathcal{ALCI}_{\mathcal{RCC}5}$, since we cannot distinguish a role from its transitive orbit (or closure) in $\mathcal{ALCI}_{\mathcal{RCC}5}$. For the same reason, an encoding of the counter concept fails in languages like $\mathcal{ALC}_{\mathcal{R}^+}$ that only provide *transitively closed* roles.

However, we did not prove EXPTIME-hardness yet. The existence of concepts which can only be satisfied in models of exponential size does *not* imply that the logic can no longer be in PSPACE (consider the $\mathcal{ALCI}_{\mathcal{RCC}5}$ model given above, which was also exponential). However, by reducing the EXPTIME-complete *two person corridor tiling game* to $\mathcal{ALCI}_{\mathcal{RCC}8}$ concept satisfiability, we can show that concept satisfiability of $\mathcal{ALCI}_{\mathcal{RCC}8}$ is indeed EXPTIME-hard. Note that this is a lower complexity-bound - $\mathcal{ALCI}_{\mathcal{RCC}8}$ might very well be even undecidable, we don't know yet. The reduction works basically like for PDL (see [7], Page 397–403), and we omit it here. The key-ingredients for a successful reduction of the two person corridor tiling game to concept satisfiability is the availability of a role and its transitive closure or orbit. We can easily exploit our already given *TPPI*/*NTPPI*-construction for this purpose. Again we will need to show how to enforce *tree models* in $\mathcal{ALCI}_{\mathcal{RCC}8}$; we do so by adding $\forall R_*.((\forall EC.\bot) \sqcap (\forall PO.\bot))$ as additional conjunct, which forbids the *PO* and *EC* relationships to hold in the models. Models are now trees; children (*TPPI* successors) can be distinguished from arbitrary offsprings (*NTPPI* successors) and siblings (*DC* successors), and there are no *PO* and *EC* successors. Exploiting this tree structure, it is not too hard to adopt the PDL EXPTIME-hardness proof in [7, pp. 398–401]. The "*PO*/*EC*-free fragment" of $\mathcal{ALCI}_{\mathcal{RCC}8}$ should be in EXPTIME - we conjecture that this fragment is EXPTIME-complete.

4.4.3 $\mathcal{ALCI}_{\mathcal{RCC}8}$ – Close to Undecidability?

We can show that $\mathcal{ALCI}_{\mathcal{RCC}8}$ is somehow "close" to being undecidable. Slight extensions would make the logic undecidable. We do not now yet if $\mathcal{ALCI}_{\mathcal{RCC}8}$ really *is* undecidable. This remains an open problem - like decidability of $\mathcal{ALCI}_{\mathcal{RCC}5}$. Note that the given complexity bounds are *lower bounds*. As long as we have no tight upper bounds we cannot say if the logics are decidable.

A classical undecidable problem is the so-called *Domino Problem*:

Definition 6 (Domino System) A domino system $\mathcal{DOM}$ is a triple $(\mathcal{D}, \mathcal{H}, \mathcal{V})$, where $\mathcal{D} = \{d_1, \ldots, d_n\}$ is a non-empty set of so-called *domino types*, $\mathcal{V} \subseteq \mathcal{D} \times \mathcal{D}$ is the vertical matching relation, and $\mathcal{H} \subseteq \mathcal{D} \times \mathcal{D}$ is the horizontal matching relation.

A *solution* of a domino system is a function $\mathsf{f} : \mathbb{N} \times \mathbb{N} \to \mathcal{D}$ (in the following we assume that $0 \in \mathbb{N}$) such that the matching relationships of the domino types are respected, i.e. for all $(i, j) \in \mathbb{N} \times \mathbb{N}$: $(\mathsf{f}(i, j), \mathsf{f}(i + 1, j)) \in \mathcal{H}$ and $(\mathsf{f}(i, j), \mathsf{f}(i, j + 1)) \in \mathcal{V}$. ■

A solution of a domino system can be seen as a tiling of the first quadrant of the plane with rectangular domino types. Each domino type can be used infinitely often during the tiling process, but the vertical and horizontal matching conditions must always be obeyed. Rotation of domino types is not allowed. Now, it is undecidable whether a domino system has a solution; i.e. there can be no algorithm that tells us whether an arbitrary domino system $\mathcal{DOM}$ can be used to tile the first quadrant of the infinite plane.

In order to *reduce* the domino problem to concept satisfiability in $\mathcal{ALCI}_{\mathcal{RCC}8}$ we would have to construct, from a given domino system $\mathcal{DOM}$, a concept $C_{\mathcal{DOM}}$ such that $C_{\mathcal{DOM}}$ is satisfiable iff $\mathcal{DOM}$ has a solution. Does there exists an encoding of the domino problem in $\mathcal{ALCI}_{\mathcal{RCC}8}$, thus proving the undecidability of $\mathcal{ALCI}_{\mathcal{RCC}8}$?

One central problem one has to address during the encoding of a domino system is how to represent (and then encode in a concept) *infinite two-dimensional grids*. The grid can be seen as the field where the domino tiles are to be placed. Thus, we have to find a concept C_{grid} whose models can be seen as *representations of infinite two-dimensional grids*. The models of C_{grid} thus have to fulfill the following conditions (see, for example, also [12], [2]):

- *Nodes* in the model represent field-positions in the grid.
- Each node has exactly one horizontal and exactly one vertical successor node, representing its adjacent fields "to the east and to the north". That is, there exists two total functions $R_X^{\mathcal{I}}$ and $R_Y^{\mathcal{I}}$ that can be extracted unambiguously from the models of C_{grid}, and each node can "access" its neighborhood by means of a construction in the language (e.g., if $R_X^{\mathcal{I}}$ and $R_Y^{\mathcal{I}}$ are roles in a DL, we can "access" them using universal value restrictions of the form $\forall R_X.\ldots$ and $\forall R_Y.\ldots$).
- The northern successor of the eastern successor and the eastern successor of the northern successor of each node must *coincide:* $R_X^{\mathcal{I}} \circ R_Y^{\mathcal{I}} = R_Y^{\mathcal{I}} \circ R_X^{\mathcal{I}}$. In our case, this condition turns out to be the most difficult one to fulfill.

Once a formula that enforces the infinite grid has been found, we can use a global axiom (GCI) of the form

$$\begin{aligned}\top \sqsubseteq\ & \sqcup_{D_i \in \mathcal{D}}(D_i \sqcap (\sqcap_{D_j \in \mathcal{D}, D_i \neq D_j} \neg D_j)) \sqcap \\ & \sqcap_{D_i \in \mathcal{D}}(D_i \to (\ \forall R_X.(\sqcup_{(D_i,D_j) \in \mathcal{H}} D_j) \sqcap \\ & \qquad\qquad \forall R_Y.(\sqcup_{(D_i,D_j) \in \mathcal{V}} D_j)))\end{aligned}$$

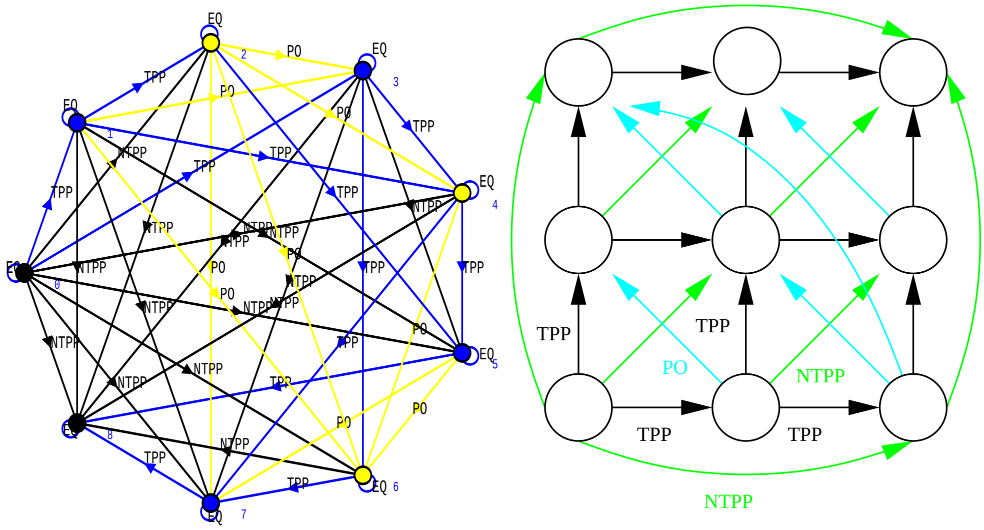


Figure 10: Frame and isomorphic 3×3 grid (some edges not drawn). The RCC8 network can be extended to grids of arbitrary size $(n \times n)$ without becoming inconsistent.

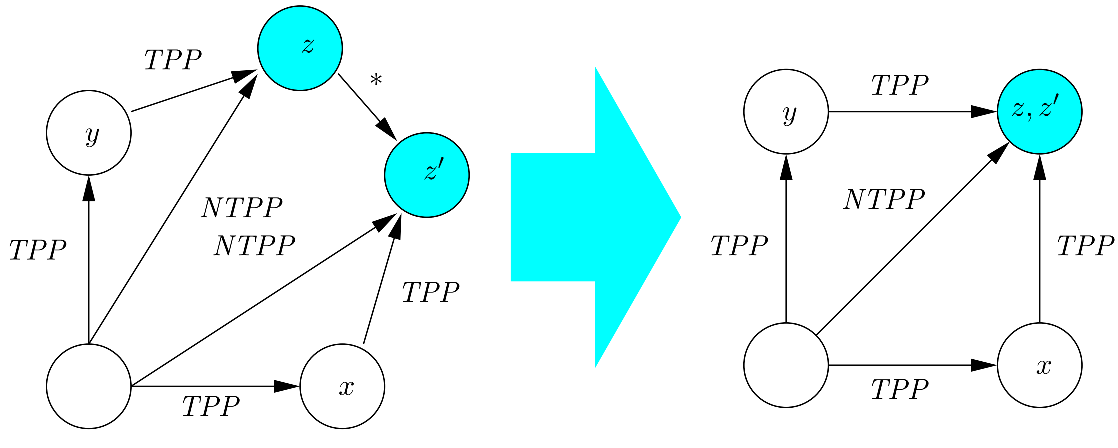
to enforce a correct tiling of the grid (recall that TBoxes can be internalized, i.e. transformed into a concept term). Please note that R_X and R_Y might not be present for direct use as roles. For example, it might be the case that only one role, say R, is present. But then, we could use additional concept names as labelings, say X and Y, and the matching condition could be

$$\sqcap_{D_i \in \mathcal{D}}(D_i \rightarrow (\ \forall R.(X \rightarrow \sqcup_{(D_i,D_j)\in\mathcal{H}} D_j) \sqcap \\ \forall R.(Y \rightarrow \sqcup_{(D_i,D_j)\in\mathcal{V}} D_j))).$$

Of course, now the presence of exactly one X-labeled R-successor and the presence of exactly one Y-labeled R-successor must be taken for granted, and it might also be more difficult to enforce the mentioned coincidence, since we have to take care not to "merge" the wrong R-successors together.

Now, what could be potential candidates for R_x and R_Y? Considering the models of $\mathcal{ALCI}_{\mathcal{RCC}8}$, one observes that they do not look like grids at all. Since every node is connected to every other node, we have to find a way how to distinguish the direct (horizontal and vertical) successors in the models from the other successors, i.e. we have to establish some kind of order on the nodes if we want to embed the grid. In the one-dimensional case, we already know how to do this: consider the models of $infinite\_even\_odd\_chain$, where $((TPPI^{\mathcal{I}})^{+} - TPPI^{\mathcal{I}}) \subseteq NTPPI^{\mathcal{I}}$ holds. Can this "schema" be extended to the two-dimensional case?

It is in fact possible to construct models of arbitrary (infinite) size that satisfy the required $\mathcal{ALCI}_{\mathcal{RCC}8}$ frame conditions, and these models are *isomorphic* to grids that satisfy the required listed three "domino encoding' conditions". A 3×3 grid is depicted in Figure 10. The depicted construction-schema can be used to construct infinite models that are isomorphic to infinite two-dimensional grids. Each node has access



$* =_{def} \{PO, EQ, TPP, NTPP, TPPI, NTPPI\}$

Figure 11: Tightening the grid?

to two *distinguished ("direct") TPP-successors:* its vertical and horizontal direct successor. In order to distingush them, i.e. to extract $R_X^{\mathcal{I}}$ and $R_Y^{\mathcal{I}}$ from the models such that $R_X^{\mathcal{I}} \cup R_Y^{\mathcal{I}} = TPP^{\mathcal{I}}$, we could use additional labeling information in the nodes (for example, like on the Pages 10 and 11 in [2]).

Even though these infinite models do exists, we have found no way to *enforce* the required depicted model(s) by means of a concept term. However, any logic that has the expressivity to enforce the depicted model by means of a formula will be undecidable. From a modal logic perspective, the satisfiability problem *relative to this class of frames* is of course undecidable. Considering the modal logic concept of *frame definiability* (exploiting the concept of frame condition axiomatizability, see below), the class of frames is of easy to enforce *on the level of frames* if we exploit additional axioms. However, this changes the logic. We do not know yet how to enforce the existence of these grids *on the level of models.*

Summing up, we encounter the following problems when trying to enforce the depicted grid structures on the level of models:

- We cannot express that each node should have *exactly two* TPP-successors. However, this is not a severe problem, since we can enforce that each node must have *at least* two different TPP-successors, which would be sufficent for the encoding.

- Suppose we can distinguish between TPP-successors representing horizontal and TPP-successors representing vertical successors (we already noted that a distinction of the successors is possible by using concept names as additional labels for the sucessors). Let's call these relations $TPP|X$ and $TPP|Y$. Then we stil have the problem that we cannot enforce the conincidence of $(TPP|X) \circ (TPP|Y)$ and $(TPP|Y) \circ (TPP|X)$. Consider Figure 11. The possible edges between z and z' are $\{PO, EQ, TPP, NTPP, TPPI, NTPPI\}$. Relying on the strong EQ-semantics, is there a way to enforce $EQ(z, z')$ and thus $z = z'$, closing the "gap" between z and z? Even though we can exploit nominals to enforce that $z = z'$, this identification of $(TPP|X) \circ (TPP|Y)$ and $(TPP|Y) \circ (TPP|X)$ works only for a *finite number of successors*, since we can only exploit a finite number of nominals for this task (the formula encoding the domino problem must be finite, of course). Adding hybrid logic's *binding operator* "$\downarrow$" ([7, pp. 444]) would change this immediately, of course. The resulting logic would be undecidable. So it is very clear to us that any additional DL constructor allowing to enforce the coincidence of z and z' (on the level of models) is making $\mathcal{ALCI}_{\mathcal{RCC}8}$ undecidable.

5 Further Issues

In order to shed some light on the remaining problems we charaterize the considered DLs in the $\mathcal{ALCI}_{\mathcal{RCC}}$ family as modal logics and try to axiomatize them in a modal logician's style.

We pinpoint some other open ideas which have to be investigated more thoroughly in the future - for example, how can we get restricted variants or fragments of the $\mathcal{ALCI}_{\mathcal{RCC}5}$ and $\mathcal{ALCI}_{\mathcal{RCC}8}$ logics which can be shown more easliy to be decidable and which are still useful in spatial reasoning scenarios? What about finite model reasoning, which is especially useful in a deductive GIS context? This part is mainly about open problems and vague ideas, but nevertheless we felt that these ideas were worthy enough to get conserved in this report.

5.1 $\mathcal{ALCI}_{\mathcal{RCC}}$ as a Modal Logic?

What can we say about $\mathcal{ALCI}_{\mathcal{RCC}}$ from a *modal logic point of view?* What about axiomatizability? Which insights do we gain by adopting a modal logician's point of view on the problems? To investigate what we can exploit from the modal logics community, we try to axiomatize the considered $\mathcal{ALCI}_{\mathcal{RCC}}$ family and look what we gain from that.

In the following we feel free to use standard modal logic notions without introducing them formally. They can all be found in [7]. First, for the syntax of concepts:

1. The concept names $\mathcal{N}_{\mathcal{C}}$ correspond to the set of proposition letters Φ of the modal logic. We will simply assume that $\Phi =_{def} \mathcal{N}_{\mathcal{C}}$.

2. The role names $\mathcal{N}_{\mathcal{R}}$ correspond to the *different modalities* of the modal logic. Each role name gives rise to one pair of *modal operators,* e.g. for $R \in \mathcal{N}_{\mathcal{R}}$ we have the binary modal operators $\Diamond_R$ and $\Box_R$. $\mathcal{ALCI}_{\mathcal{RCC}}$ is a *multi-modal* logic.

3. Using the "Schild translation" ([24]) Φ, we can translate any $\mathcal{ALCI}_{\mathcal{RCC}}$ concept C into a modal formula $\phi(C)$: $\phi(CN) =_{def} CN$ for $CN \in \mathcal{N}_{\mathcal{C}}$, and $\phi(\neg C) =_{def}$

$\neg\phi(C)$, $\phi(C \sqcap D) =_{def} \phi(C) \wedge \phi(D)$, $\phi(C \sqcup D) =_{def} \phi(C) \vee \phi(D)$, $\phi(\exists R.C) =_{def} \Diamond_R \phi(C)$ and $\phi(\forall R.C) =_{def} \Box_R \phi(C)$.

For example, $\phi(\exists PP.C \sqcap \forall PP.\exists PP.C) = \Diamond_{PP}\ C \wedge \Box_{PP}\ \Diamond_{PP}\ C$. However, this is only a syntactic transformation. Without any additional "frame conditions" on the interpretations of the roles (which correspond to the accessibility relations in the Kripke models) we would only get the basic normal multi-modal logic K_m. K_m is just a set of formulas, defined axiomatically:

- All instances of propositional tautologies
- (K) $\Box_R(p \rightarrow q) \rightarrow (\Box_R p \rightarrow \Box_R q)$, for all $R \in \mathcal{N_R}$
- (Dual) $\Diamond_R p \leftrightarrow \neg\Box_R \neg p$, for all $R \in \mathcal{N_R}$

Note that p and q might be substituted by arbitrary formulas. The set if closed under uniform substitutions, *generalization* (given ϕ, prove $\Box_R \phi$) and applications of *modus ponens.* K_M is a very weak modal logic, which does not capture the requirements that we want to impose on the interpretations of the roles. But we can enforce these so-called *frame conditions* by adding appropriate *frame axioms,* exploiting the second-order concept of *frame definability* resp. *frame validity of axioms.* The axioms should be sound and complete w.r.t. the intended class of frames.

Definition 7 A *Kripke frame* for a multi-modal logic with $\mathcal{N_R} = \{R_1, \ldots, R_n\}$ is a tuple $\mathcal{F} =_{def} < \Delta^\mathcal{I}, R_1^\mathcal{I}, \ldots, R_n^\mathcal{I} >$. The $R_i^\mathcal{I}$ are binary relations on $\Delta^\mathcal{I}$, called accessibility relations. A *Kripke model* is a frame $\mathcal{F}$ for which we additionally give interpretations for the concept names: $\mathcal{M} =_{def} < \mathcal{F}, CN_1^\mathcal{I}, \ldots, CN_n^\mathcal{I} >$ (also written $<\mathcal{F}, V>$, with V being the valuation function). The model $\mathcal{M}$ is said to by *based* on the frame $\mathcal{F}$.

A formula ϕ is called satisfiable iff there is a model $\mathcal{M}$ and a point $m \in \Delta^\mathcal{I}$ such that ϕ evaluates to true at this point, denoted by $\mathcal{M}, m \models \phi$. The satisfaction relationship "$\models$" is defined as for the DL $\mathcal{ALCI_{RCC}}$. The *global satisfiability problem* asks for a model such that the formula is satisfied *at all points.* Often we are not interested in arbitrary models, but in *models based on certain frames from a class of frames.* This is the *relative* satisfiability problem (relative to a class of frames). In case of $\mathcal{ALCI_{RCC}}$, we are not interested in arbitrary satisfiability, but in satisfiability relative to the class of $\mathcal{ALCI_{RCC}}$ frames which satisfy the mentioned frame conditions.

A formula is *valid on a certain frame* $\mathcal{F}$ iff for all models *based on the frame* $\mathcal{F}$ - this means the models only differ in the interpretations for the concept names - the formula is globally satisfied. A formula is *valid on a class of frames* iff it is valid on all frames in that class. ■

Now it happens to be the case that sets of formulas, defined by frame axioms, can define *classes of frames* - the set of frames on which the frame axioms are valid. A *sound and complete axiomatization* for a class of frames characterizes this class utilizing frame axioms such that the set of frames on which the axioms are valid is exactly the class of wanted frames. In our case, we would like to find a sound and complete axiomatization for the class of $\mathcal{ALCI}_{\mathcal{RCC}}$ frames. Can we? We proceed as follows:

1. If we add, for all $R \in \mathcal{N_R}$, the axiom schemas $p \rightarrow \Box_R \Diamond_{\mathsf{inv}(R)}\ p$ and $p \rightarrow \Box_{\mathsf{inv}(R)} \Diamond_R\ p$, we will have the required *converse relationships.* The schema is valid exactly on the class of frames where $R^{\mathcal{I}} = (\mathsf{inv}(R)^{\mathcal{I}})^{-1}$, for all $R \in \mathcal{N_R}$. The schema is know as "multi-modal" (B).

2. If we add, for all $S \circ T \sqsubseteq R_1 \sqcup \cdots \sqcup R_n$ according to the appropriate RCC composition table, the axiom schema $\Diamond_S \Diamond_T\ p \rightarrow \Diamond_{R_1}\ p \vee \cdots \vee \Diamond_{R_n}\ p$, this axiomatizes the *composition requirements.* The schema could be called "generalized K4".

3. Under the weak EQ semantics, the role EQ could be axiomatized as an "$S5$-like" modality; we only have to add the (T)-axiom: $p \rightarrow \Diamond_{EQ}\ p$. Note that $\Diamond_{EQ} \Diamond_{EQ}\ p \rightarrow \Diamond_{EQ}\ p$ and $p \rightarrow \Box_{EQ} \Diamond_{EQ}\ p$ have already been added. However, there is no corresponding axiom schema if we want to employ the *strong* EQ semantics.

4. What about disjointness of roles? Unfortunately, disjointness of roles is *not modally definable,* i.e., there is no set of axioms to define the class of disjoint frames that we need. We have left the realm of standard modal logics and will need an *extended* modal logic, see below.

5. What about the "one cluster" requirement? The models enforced so far might have the form of several, *separated* clusters (due to the "generalized K4" axioms). It is known that the *universal modality* R_* is not modally definable in the basic modal language, because modal satisfaction is invariant *under disjoint union of models.* However, considering satisfaction of formulas only, e.g. of ϕ at a point i in a model, we can simply take its *generated sub-model* around i (effectively discarding all non-accessible parts from the model) and then use $R_1 \vee \cdots \vee R_n$ as the global modality in the generated sub model (of course, this is no longer true if we consider ABoxes, etc.). We therefore consider, in the following, only the class of *connected frames*, which is, due to the totality of the composition axioms, equivalent to the class of *one cluster frames*; in this class, the global modality is indeed definable as we did here.

What have we achieved so far? The axiomatization is sound and complete w.r.t. a class of models/frames which satisfy *most* of the required frame properties: EQ is interpreted as an equivalence relation, the compositional axioms from the RCC table are satisfied, as well as the converse relationships. Only the disjointness requirement is not satisfied. Note that irreflexiveness, SPO- ness of $PP^{\mathcal{I}}$ and $PPI^{\mathcal{I}}$ are all consequences of the disjointness requirement. Not surprisingly, irreflexiveness isn't modally definable as well. One idea might be to take a "pre-model" of ϕ which does not satisfy the disjointness requirement, and transform it into a "real" model, by performing some kind of standard modal logic "model surgery" (filtratation, unwinding, ...) in order to satisfy the outstanding frame conditions. Unfortunately, this doesn't work, since ϕ might be satisfiable in the pre-model, but will have no "real" model at all (see Subsection 2.1).

In order to axiomatize the *remaining problematic* frame conditions we use basic *hybrid* modal logic. This logic offers nominals which are denoted like proposition letters (e.g., i). Nominals are simply *names* for states and are interpreted as singletons (similar to ABox individuals). Additionally, basic hybrid modal logic provides the satisfaction operator, $@i$. Intuitevely, $@i\phi$ means that the state named i in the model must satisfy ϕ: $\mathcal{M}, i \models \phi$. The following frame axioms would enforce the required missing frame properties:

1. Disjointness requirement: $\forall R \in \mathcal{N_R} : @\ i\Diamond_R\ j \rightarrow @\ i \bigwedge_{S \in \mathcal{N_R} \setminus R} \neg\Diamond_S\ j$
2. Strong EQ-semantics: $\neg @\ i\Diamond_{EQ} j$
3. The "one cluster" requirement: $\bigvee_{R \in \mathcal{N_R}} @\ i\Diamond_R j$

A *pure formula* is a hybrid formula that does not mention propositional letters. All given axioms are pure. In [7, pp. 437] it is shown that *adding a set of pure axioms* to the basic *hybrid* modal logic K (which is equipped with an augmented proof system and additional axioms for the nominals and @ etc. which we do not want to repeat here) automatically produces a logic which is *sound and complete w.r.t. the intended class of frames.* Since we've already left K by the addition of the other axioms above, we need to rewrite the already given axioms as pure formulas as well, which is easy. We then get the following sound an complete axiomatic system for $\mathcal{ALCI}_{\mathcal{RCC}}$:

1. "One cluster": $\bigvee_{R \in \mathcal{N_R}} @\ i\Diamond_R j$
2. Strong EQ: $\neg @\ i\Diamond_{EQ} j$
3. Strong EQ: $i \rightarrow \Diamond_{EQ} i$
4. Disjointness: $\forall R \in \mathcal{N_R} : @\ i\Diamond_R\ j \rightarrow @\ i \bigwedge_{S \in \mathcal{N_R} \setminus R} \neg\Diamond_S\ j$
5. Converses: $\forall R \in \mathcal{N_R} : @\ i\Diamond_R\ j \rightarrow @\ j\Diamond_{\mathsf{inv}(R)}\ i$
6. Compositions: for all $S \circ T \sqsubseteq R_1 \sqcup \cdots \sqcup R_n$ from the corresponding RCC composition table, add $@\ i\Diamond_S\ \Diamond_T\ j \rightarrow @\ i\Diamond_{R_1} j \vee \cdots \vee @\ i\Diamond_{R_n} j$

Unfortunately a sound and complete axiom schema for $\mathcal{ALCI}_{\mathcal{RCC}}$ does not help us in establishing decidability, since the logics do not have the finite model property. However, the axiomatic system is nevertheless a good starting point for future research, and contributes to a question raised by Cohn in [8] concerning the completeness of the "RCC8 modal logic" proposed there (however, the logics are only similar, not identical).

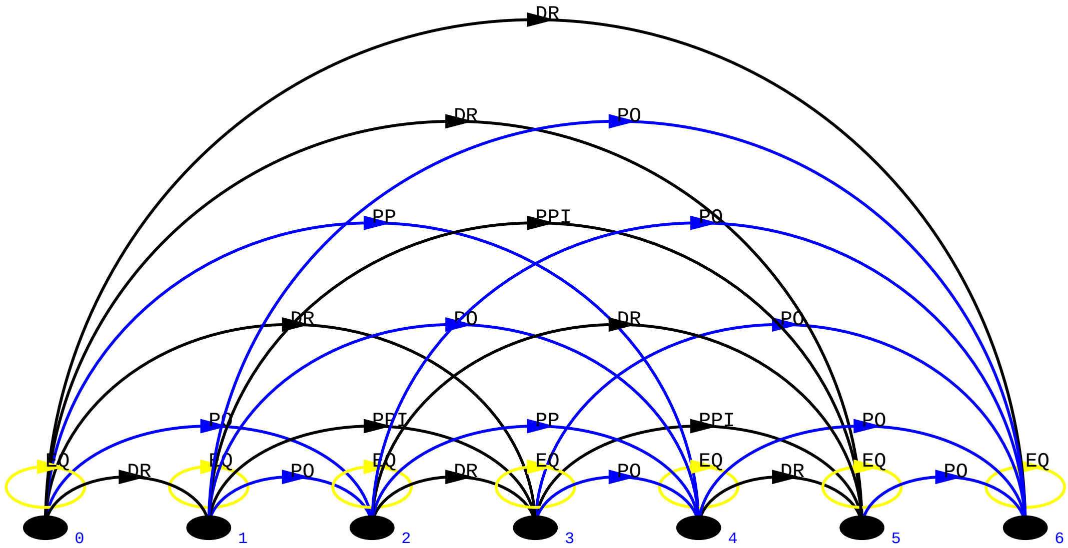


Figure 12: Part of an infinite model, enforced by a concept term that doesn't mention PP and/or PPI: $\exists PO.((\exists DR.C) \sqcap (\forall PO.\neg C) \sqcap (\forall DR.\neg C))$ enforces to chose PPI from $PO \circ DR \sqsubseteq DR \sqcup PO \sqcup PPI$, and $DR \circ PPI \sqsubseteq DR$. Due to the alternating labeling of the nodes with C and $\neg C$ and the "determinism" caused by $DR \circ PPI \sqsubseteq DR$ and $\forall DR. \ldots$ there cannot be a finite model (the sceptical reader might try to construct one anyway, but will not have success).

5.2 Finite Model Reasoning with $\mathcal{ALCI}_{\mathcal{RCC}5}$?

If we have such problems showing decidability or undecidability of $\mathcal{ALCI}_{\mathcal{RCC}5}$ and $\mathcal{ALCI}_{\mathcal{RCC}8}$, then might it probably be easier to restrict ourselves to *finite model reasoning*, i.e. impose a semantics where concepts that have only infinite models are considered as *unsatisfiable?* This would also be appealing from an application point of view, since there is not much practical use of concepts that only allow infinite models, since we want to model real world spatial phenomena on a qualitative level.

Generally speaking, finite model reasoning is not necessarily easier than reasoning with general models (Trakhtenbrot). However, it *may be easier.* For example,

- $(\exists PP.\top) \sqcap (\forall PP.\exists PP.\top)$,
- $(\exists DR.\top) \sqcap (\forall DR.\exists PPI.\top)$, and
- $(\exists DR.\exists PO.C) \sqcap (\forall PO.\neg C) \sqcap (\forall DR.\neg C) \sqcap$
  $(\forall PP.((\exists DR.\exists PO.C) \sqcap (\forall PO.\neg C) \sqcap (\forall DR.\neg C)))$

only have infinite models. All example concepts use the PP- and/or PPI-role. It is indeed tempting to suspect that the use of PP and/or PPI within universal and/or existential value restrictions is responsible for spawning the infinite models.

Unfortunately, this seemingly plausible idea, as formulated in [27], is wrong. There are indeed concepts that do not even mention PP and/or PPI, but still do not have finite models. Consider

$$\begin{aligned}&\exists DR.\top \sqcap \\ &\forall DR.(\ \exists PO.\exists DR.C \sqcap \\ &\qquad \forall PO.\neg C \sqcap \\ &\qquad \forall DR.\neg C).\end{aligned}$$

A (part of an infinite) model of this term is depicted in Figure 12. In every model of this concept, the PPI edges bridging $PO \circ DR$ and the DR edges originating from the root node 0 must be present, as well as the $DR \circ PO \circ DR \circ PO \circ \ldots$-pattern "at the bottom of the model".

However, inspecting the models of the concept (all of which are infinite!), we can somehow still say that PPI is "responsible" for spawning the infinite model, in combination with $DR \circ PPI \sqsubseteq DR$ and $\forall DR. \ldots$. Unfortunately, this cannot be detected on a syntactical level. But there is some hope that a tableau calculus could detect the enforced presence of PP and/or PPI, e.g. by adding some form of "self introspection" to the rules of a tableau calculus. However, this has not been worked out yet. We were experimenting with some kind of "infinity checker" that tried to detect whenever an infinite structure is enforced by the current tableau expansion history (of course, this might be undecidable as well).

An *infinity checker* is somehow similar to a *blocking condition.* A *blocking condition* (see, for example, [16], [17]) is needed for tableau calculi for logics like $\mathcal{ALC}_{\mathcal{R}^+}$, which is plain $\mathcal{ALC}$ with additional transitively closed roles (see [23]). Suppose that PP is simply a transitively closed role in $\mathcal{ALC}_{\mathcal{R}^+}$: then, a naive $\mathcal{ALC}_{\mathcal{R}^+}$ calculus would not terminate for concepts like $(\exists PP.\top) \sqcap (\forall PP.\exists PP.\top)$. For the same reason, an $\mathcal{ALCI}_{\mathcal{RCC}5}$ calculus would not terminate - the calculus would try to construct an infinite number of PP-successor nodes. Generally speaking, the tableau expansion process can be seen as an attempt to build a *finite pseudo-model* of the concept under consideration. In order to prevent the calculus from infinite expansions, the blocking condition *blocks* the generation of new PP-successor nodes at same point in the expansion process. The blocking condition then has to ensure that, whenever it blocks the expansion of the calculus, that from this so-far constructed finite pseudo-model a (possibly infinite) model can always be constructed. In the case of $\mathcal{ALC}_{\mathcal{R}^+}$, even a finite model can be constructed from the blocked pseudo-model. This is obviously not the case for $\mathcal{ALCI}_{\mathcal{RCC}5}$ and $\mathcal{ALCI}_{\mathcal{RCC}8}$. An infinity checker is in fact a *stronger* predicate than a blocking condition - the blocking condition would have to ensure that whenever it returns TRUE, an infinite *model* can be constructed. In contrast, the infinity checker must not know whether the concept under consideration is satisfiable in the infinite or not. The problem we are trying to solve seems to be related to the well-foundedness problem in *part-whole-reasoning.*

Being even more restrictive, we can only consider finite models with a fixed maximal cardinality, say n. We have already shown that we can enforce "maximal cardinality reasoning" with nominals, effectively ruling out all models with $|\Delta^{\mathcal{I}}| \geq n$ for some fixed n. If we do so, we will have a decidable DL (for $\mathcal{ALCI}_{\mathcal{RCC}5}$ and $\mathcal{ALCI}_{\mathcal{RCC}8}$). This is due to the fact that a logic with a sound and complete axiomatization (which is given above) which also has the finite model property is decidable.

5.3 Restricted Decidable Fragments of $\mathcal{ALCI}_{\mathcal{RCC}5}$ and $\mathcal{ALCI}_{\mathcal{RCC}8}$?

What else can we do? One possibility is to identify fragments of $\mathcal{ALCI}_{\mathcal{RCC}5}$ and/or $\mathcal{ALCI}_{\mathcal{RCC}8}$ whose decidability can easily be shown. For example, a certain *syntax-restriction criterion* in the logic $\mathcal{ALCRP}(\mathcal{D})$ ([15, 14, 20]) brings back the finite model property, which was already lost in general unrestricted $\mathcal{ALCRP}(\mathcal{D})$, thus ensuring the decidability of its syntax-restricted variant. A similar syntax-restriction, derived from restricted $\mathcal{ALCRP}(\mathcal{D})$, can also be formulated for $\mathcal{ALCI}_{\mathcal{RCC}5}$ and $\mathcal{ALCI}_{\mathcal{RCC}8}$. Laxly speaking, we would have to reject every $\mathcal{ALCI}_{\mathcal{RCC}5}$- and/or $\mathcal{ALCI}_{\mathcal{RCC}8}$-concept showing quantifier patterns of the form "$\exists R. \ldots \forall S. \ldots$" and/or "$\forall R. \ldots \exists S. \ldots$" in its negation normal form. Referring to the TBox in our deductive GIS example, we can then no longer model the concept *hamburg* as given there. Since *hamburg* violates the syntax-restriction criterion which is derived from $\mathcal{ALCRP}(\mathcal{D})$, the concept *hamburg* could not be modeled in $\mathcal{ALCRP}(\mathcal{S}_2)$ neither. $\mathcal{ALCRP}(\mathcal{S}_2)$ is a *specialization* of $\mathcal{ALCRP}(\mathcal{D})$, offering a specific *concrete domain* $\mathcal{D} = \mathcal{S}_2$ which could be used in a similar way like $\mathcal{ALCI}_{\mathcal{RCC}8}$ to address qualitative spatial reasoning tasks.

An other more-or-less obvious idea of how to get a restricted decidable logic is to *allow only purely boolean concepts within universal value restrictions in its NNFs.* That is, if D is the concept to be tested for satisfiability, then for all $\forall R.C \in \mathsf{sub}(D)$, C must be a boolean concept (which means that there is no $\forall S.E \in \mathsf{sub}(C)$ and no $\exists S.E \in \mathsf{sub}(C)$ for any R, E).

However, all kinds of syntax-restrictions make modeling quite hard: first of all, the syntax-restrictions need to be checked at the level of the NNF of a concept, and the NNF of a concept can be quite different from the original formulation of that concept. Inference tasks like computation of subsumption relationships between concepts will typically be solved by reduction to concept satisfiability: to check whether $C \dot{\sqsubseteq} D$ holds a DL-system would typically check for unsatisfiability of the concept $C \sqcap \neg D$. However, then not only D, but also $\neg D$ must be syntax restricted, as well as $C \sqcap \neg D$, and so on. Obviously the syntax restriction criterion must be *preserved under negation.* Negating a syntax-restricted concept must again yield a syntax-restricted concept.

In contrast, consider the syntax-restriction which only allows for boolean concepts as qualifications within universal value restrictions. This criterion is obviously not preserved under negation. If we want to turn into into a negation-preserved criterion we would also have to forbid non-boolean concepts within the qualifications of *existential value restrictions* as well. But then this gives us exactly the syntax-restriction criterion which we have already derived from $\mathcal{ALCRP}(\mathcal{D})$.

6 Summary and Outlook

Summing up, we have made some first steps from the general, previously considered $\mathcal{ALC}_{\mathcal{RA}^\ominus}$, $\mathcal{ALC}_{\mathcal{RA}}$ and $\mathcal{ALC}_{\mathcal{RASG}}$ DLs to languages which are more useful for qualitative spatial reasoning tasks. However, there are still many open questions which need to be solved in the future. We've already highlighted some open points for future research in the previous sections.

References

[1] F. Baader, D. McGuinness, D. Nardi, and P.P. Schneider, editors. *The Description Logic Handbook – Theory, Implementation and Applications.* Cambridge University Press, 1. edition, 2002.

[2] F. Baader and U. Sattler. Expressive number restrictions in description logics. *Journal of Logic and Computation*, 9(3):319–350, 1999.

[3] F. Baader and U. Sattler. Tableau algorithms for description logics. In R. Dyckhoff, editor, *Proceedings of the International Conference on Automated Reasoning with Tableaux and Related Methods (Tableaux 2000)*, volume 1847 of *Lecture Notes in Artificial Intelligence*, pages 1–18, St Andrews, Scotland, UK, 2000. Springer-Verlag.

[4] F. Baader and U. Sattler. An overview of tableau algorithms for description logics. *Studia Logica*, 69:5–40, 2001.

[5] B. Bennett. Spatial reasoning with propositional logics. In J. Doyle, E. Sandewall, and P. Torasso, editors, *Fourth International Conference on Principles of Knowledge Representation, Bonn, Germany, May 24-27, 1994*, pages 51–62, May 1994.

[6] B. Bennett. Modal logics for qualitative spatial reasoning. *Bull. of the IGPL*, 3:1–22, 1995.

[7] Patrick Blackburn, Maarten de Rijke, and Yde Venema. *Modal Logic.* Cambridge Tracts in Theoretical Computer Science, No 53, 2001.

[8] A. G. Cohn. Modal and Non Modal Qualitative Spatial Logics. In F.D. Anger, H.M. Guesgen, and J. van Benthem, editors, *Proceedings of the Workshop on Spatial and Temporal Reasoning, IJCAI*, 1993.

[9] A. G. Cohn, B. Bennett, J. M. Gooday, and N. Gotts. RCC: a calculus for region based qualitative spatial reasoning. *GeoInformatica*, 1:275–316, 1997.

[10] M. Fitting. Basic Modal Logic. In D. M. Gabbay, C. J. Hogger, and J. A. Robinson, editors, *Handbook of Logic in Artifical Intelligence*, volume 1, pages 368–449. Oxford University Press, 1993.

[11] E. Grädel and M. Otto. On Logics with Two Variables. *Theoretical Computer Science*, 224:73–113, 1999.

[12] E. Grädel, M. Otto, and E. Rosen. Undecidability Results for Two-Variable Logics. *Archive for Mathematical Logic*, 38:313–354, 1999. See also: Proceedings of 14th Symposium on Theoretical Aspects of Computer Science STACS'97, Lecture Notes in Computer Science No. 1200, Springer 1997, 249–260.

[13] M. Grigni, D. Papadias, and C. Papadimitriou. Topological Inference. In C. Mellish, editor, *14th International Joint Conference on Artificial Intelligence*, pages 901–906, 1995.

[14] V. Haarslev, C. Lutz, and R. Möller. Foundations of spatioterminological reasoning with description logics. In T. Cohn, L. Schubert, and S. Shapiro, editors, *Proceedings of Sixth International Conference on Principles of Knowledge Representation and Reasoning (KR'98), Trento, Italy, June 2-5, 1998*, pages 112–123, June 1998.

[15] V. Haarslev, C. Lutz, and R. Möller. A description logic with concrete domains and a role-forming predicate operator. *Journal of Logic and Computation*, 9(3):351–384, June 1999.

[16] I. Horrocks, U. Sattler, and S. Tobies. Practical reasoning for expressive description logics. In *Proceedings of the 6th International Conference on Logic for Programming and Automated Reasoning (LPAR'99)*, 1999.

[17] I. Horrocks, U. Sattler, and S. Tobies. Practical reasoning for very expressive description logics. *Logic Journal of the IGPL*, 8(3):239–264, May 2000.

[18] G. E. Hughes and M. J. Cresswell. *An Introduction to Modal Logic*. London, 1972.

[19] C. Lutz. Representation of Topological Information in Description Logics (in German). Master's thesis, University of Hamburg, Computer Science Department, February 1998.

[20] C. Lutz, V. Haarslev, and R. Möller. A concept language with role-forming predicate restrictions. Technical Report FBI-HH-M-276/97, University of Hamburg, Computer Science Department, 1997.

[21] R. Möller, V. Haarslev, and C. Lutz. Spatioterminological reasoning based on geometric inferences: The $\mathcal{ALCRP}(\mathcal{D})$ approach. Technical Report FBI-HH-M-277/97, University of Hamburg, Computer Science Department, 1997.

[22] D. A. Randell, Z. Cui, and A. G. Cohn. A Spatial Logic based on Regions and Connections. In B. Nebel, C. Rich, and W. Swartout, editors, *Principles of Knowledge Representation and Reasoning*, pages 165–176, 1992.

[23] U. Sattler. A concept language extended with different kinds of transitive roles. In G. Görz and S. Hölldobler, editors, *20. Deutsche Jahrestagung für Künstliche Intelligenz*, number 1137 in Lecture Notes in Artificial Intelligence, pages 333–345. Springer Verlag, Berlin, 1996.

[24] K. Schild. A correspondence theory for terminological logics: Preliminary report. In *Twelfth International Conference on Artificial Intelligence, Darling Harbour, Sydney, Australia, Aug. 24-30, 1991*, pages 466–471, August 1991.

[25] M. Schmidt-Schauß and G. Smolka. Attributive concept descriptions with complements. *Artifical Intelligence*, 48:1–26, 1991.

[26] M. Wessel. Obstacles on the way to spatial reasoning with description logics – some undecidability results. In Carole Goble Deborah L. McGuinness, Peter F. Patel-Schneider and Ralf M öller, editors, *Proceedings of the International Workshop on Description Logics 2001 (DL2001)*, number 49 in CEUR-WS, pages 122–131, Stanford University, California, USA, August 1-3 2001. RWTH Aachen. Proceedings online available from `http://SunSITE.Informatik.RWTH-Aachen.DE/Publications/CEUR-WS/Vol-49/`.

[27] M. Wessel. On spatial reasoning with description logics - position paper. In Sergio Tessaris Ian Horrocks, editor, *Proceedings of the International Workshop on Description Logics 2002 (DL2002)*, number 53 in CEUR-WS, pages 156–163, Toulouse, France, April 19–21 2002. RWTH Aachen. Proceedings online available from `http://SunSITE.Informatik.RWTH-Aachen.DE/Publications/CEUR-WS/Vol-53/`.

[28] M. Wessel. Decidable and undecidable extensions of $\mathcal{ALC}$ with composition-based role inclusion axioms. Technical Report FBI–HH–M–301/01, University of Hamburg, Computer Science Department, December 2000. Available at `http://kogs-www.informatik.uni-hamburg.de/~mwessel/report5.{ps.gz | pdf}`.

[29] M. Wessel. Undecidabity of $\mathcal{ALC}_{\mathcal{RA}}$. Technical Report FBI–HH–M–302/01, University of Hamburg, Computer Science Department, March 2001. Available at `http://kogs-www.informatik.uni-hamburg.de/~mwessel/report6.{ps.gz | pdf}`.

[30] M. Wessel. Obstacles on the way to spatial reasoning with description logics – undecidability of $\mathcal{ALC}_{\mathcal{RA}^\ominus}$. Technical Report FBI–HH–M–297/00, University of Hamburg, Computer Science Department, October 2000. Available at `http://kogs-www.informatik.uni-hamburg.de/~mwessel/report4.{ps.gz | pdf}`.