Global Image Properties

Global image properties refer to an image as a whole rather than components. Computation of global image properties is often required for image enhancement, preceding image analysis.

We treat

- **empirical mean and variance**
- **histograms**
- **projections**
- **cross-sections**
- **frequency spectrum**

Empirical Mean and Variance

Empirical mean = average of all pixels of an image

$$\bar{g} = \frac{1}{MN}\sum_{m=0}^{M-1}\sum_{n=0}^{N-1} g_{mn}$$ with M x N image size

Simplified notation: $$\bar{g} = \frac{1}{K}\sum_{k=0}^{K-1} g_k$$

Incremental computation: $\bar{g}_0 = 0$ $\quad \bar{g}_k = \frac{\bar{g}_{k-1}(k-1) + g_k}{k}$ $\quad$ k = 2 ... K

Empirical variance = average of squared deviation of all pixels from mean

$$\sigma^2 = \frac{1}{K}\sum_{k=1}^{K}(g_k - \bar{g})^2 = \frac{1}{K}\sum_{k=1}^{K} g_k^2 - \bar{g}^2$$

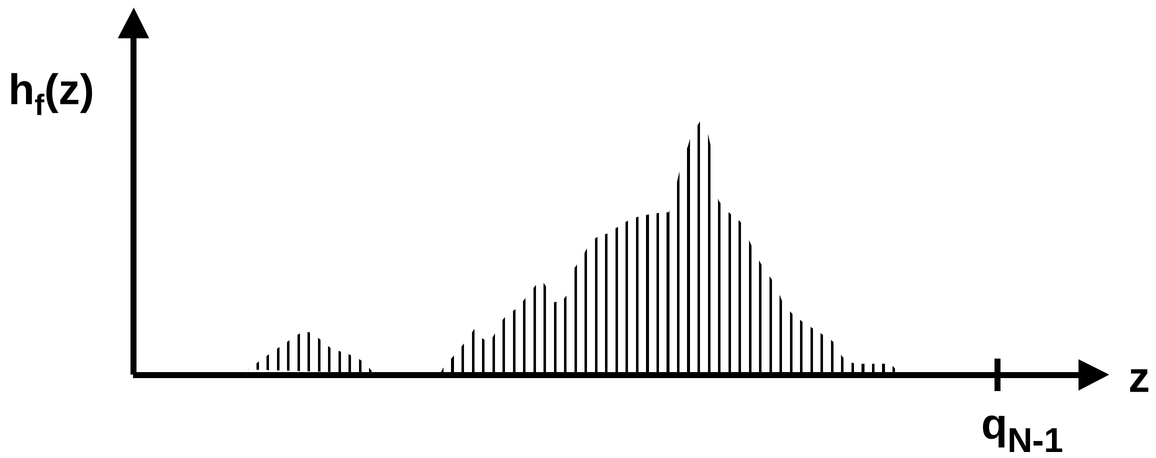
Incremental computation:

$\sigma_0^2 = 0 \quad \sigma_k^2 = \frac{(\sigma_{k-1}^2 + \bar{g}_{k-1}^2)(k-1) + g_k^2}{k} - \left(\frac{\bar{g}_{k-1}(k-1) + g_k}{k}\right)^2$ $\quad$ k = 2 ... K

Greyvalue Histograms

A greyvalue histogram $h_f(z)$ of an image f provides the frequency of greyvalues z in the image.

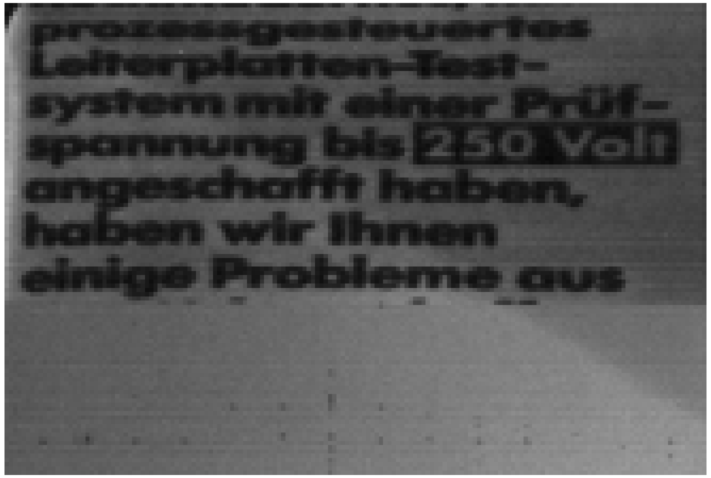
The histogram of an image with N quantization levels is represented by a 1D array mit N elements.



A greyvalue <u>histogram</u> describes discrete values, a greyvalue <u>distribution</u> describes continuous values.

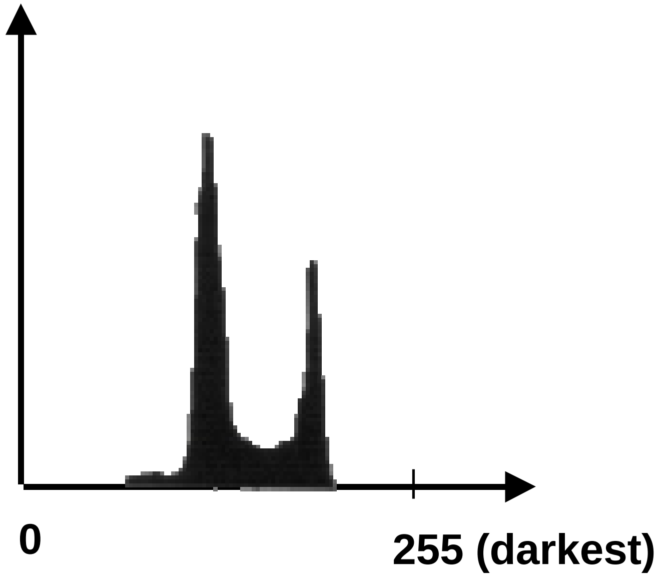
Example of Greyvalue Histogram

image

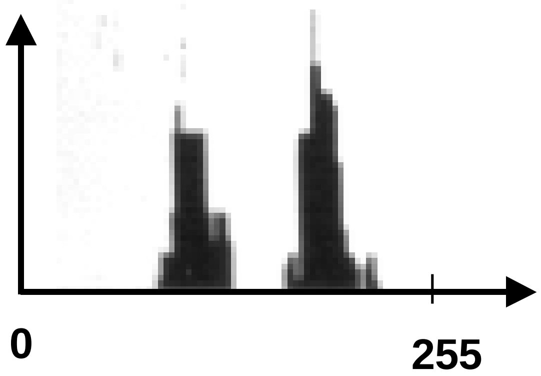


prozessgesteuertes
Leiterplatten-Test-
system mit einer Prüf-
spannung bis 250 Volt
angeschafft haben,
haben wir Ihnen
einige Probleme aus

histogram



A histogram can be "sharpened" by discounting pixels at edges (more about edges later):

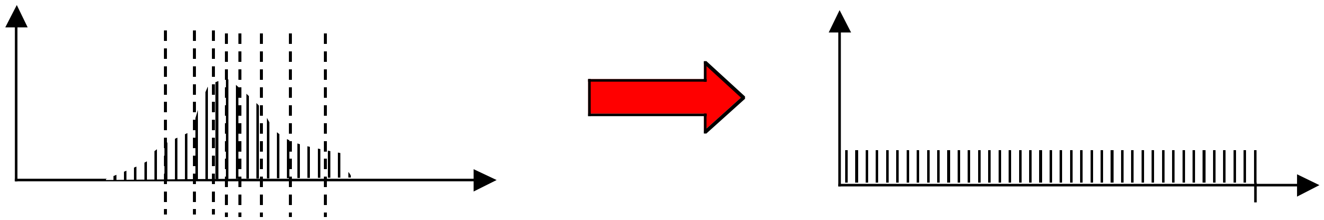


Histogram Modification

Greyvalues may be remapped into new greyvalues to

- **facilitate image analysis**
- **improve subjective image quality**

<u>Example</u>: Histogram equalization

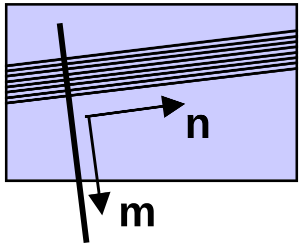


1. Cut histogram into N stripes of equal area (N = new number of greyvalues)
2. Assign new greyvalues to consecutive stripes

Examples show improved resolution of image parts with most frequent greyvalues (road surface)

Projections

A projection of greyvalues in an image is the sum of all greyvalues orthogonal to a base line:

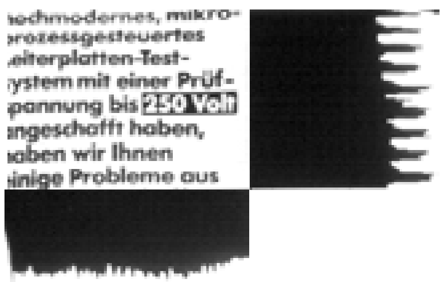


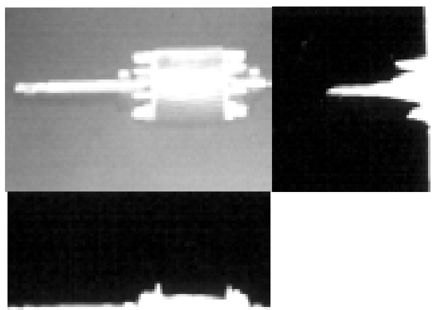
$$p_m = \sum_n g_{mn}$$

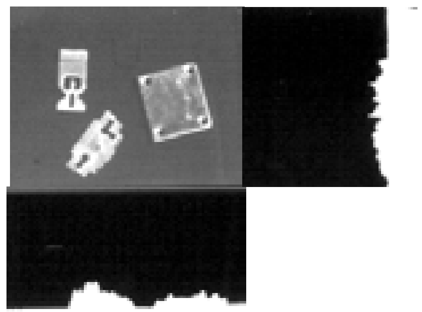
Often used:

"row profile" = row vector of all (normalized) column sums

"column profile" = column vector of all (normalized) row sums



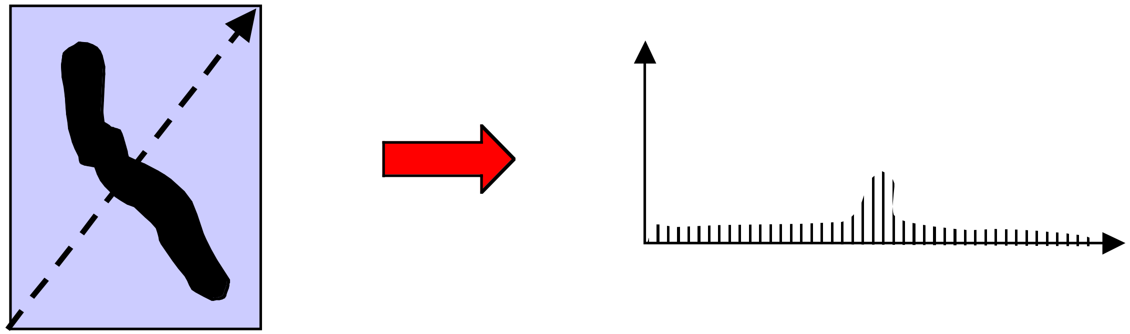




Cross-sections

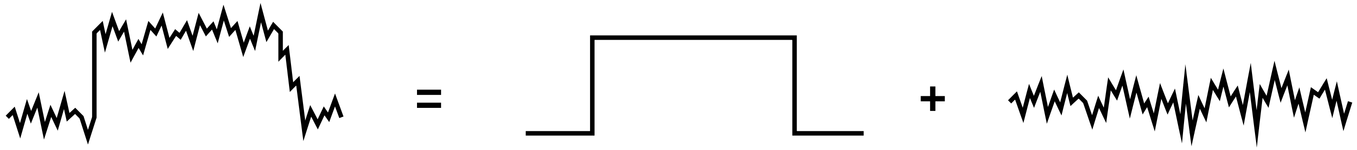
A cross-section of a greyvalue image is a vector of all pixels along a straight line through the image.

- **fast test for localizing objects**
- **commonly taken along a row or column**



Noise

Deviations from an ideal image can often be modelled as additive noise:



Typical properties:

- **mean 0, variance $\sigma^2 > 0$**
- **spatially uncorrelated: E[$r_{ij}\, r_{mn}$] = 0 for ij ≠ mn**
- **temporally uncorrelated: E[$r_{ij,t1}\, r_{ij,t2}$] = 0 for t1 ≠ t2**
- **Gaussian probability density:** $$p(r) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{r^2}{2\sigma^2}}$$

Noise arises from analog signal generation (e.g. amplification) and transmission.

There are several other noise models other than additive noise.

Noise Removal by Averaging

Principle: $\hat{r}_K = \frac{1}{K}\sum_{k=1}^{K} r_k \Rightarrow 0$ **sample mean approaches density mean**

There are basically 2 ways to "average out" noise:

- **temporal averaging if several samples $g_{ij,t}$ of the same pixel but at different times t = 1 ... T are available**
- **spatial averaging if $g_{mn} \approx g_{ij}$ for all pixels g_{mn} in a region around g_{ij}**

How effective is averaging of K greyvalues?

$$\hat{r}_K = \frac{1}{K}\sum_{k=1}^{K} r_k$$ **is random variable with mean and variance depending on K**

$$E\left[\hat{r}_K\right] = \frac{1}{K}\sum_{k=1}^{K} E\left[r_k\right] = 0$$ **mean**

$$E\left[(\hat{r}_K - E\left[\hat{r}_K\right])^2\right] = E\left[\hat{r}_K{}^2\right] = E\left[\frac{1}{K^2}\left(\sum_{k=1}^{K} r_k\right)^2\right] = \frac{1}{K^2}\sum_{k=1}^{K} E\left[r_k^2\right] = \frac{\sigma^2}{K}$$ **variance**

<u>Example</u>: In order cut the standard deviation σ in half, 4 values have to averaged

Example of Averaging

intensity averaging with 5 x 5 mask

$$\frac{1}{25}$$

1	1	1	1	1
1	1	1	1	1
1	1	1	1	1
1	1	1	1	1
1	1	1	1	1

Simple Smoothing Operations

1. Averaging

$$\hat{g}_{ij} = \frac{1}{|D|} \sum_{g_{mn} \in D} g_{mn}$$

D is region around g_{ij}

Example of
3-by-3 region D

	ij	

2. Removal of outliers

$$\hat{g}_{ij} = \begin{cases} \frac{1}{|D|} \sum_{g_{mn} \in D} g_{mn} & \text{if } \left| g_{ij} - \frac{1}{|D|} \sum_{g_{mn} \in D} g_{mn} \right| \geq S \\ g_{ij} & \end{cases}$$

S is threshold

3. Weighted average

$$\hat{g}_{ij} = \frac{1}{\sum w_k} \sum_{g_k \in D} w_k g_k$$

w_k = weights in D

Example of weights
in 3-by-3 region

1	2	1
2	3	2
1	2	1

Note that these operations are heuristics and not well founded!

Bimodal Averaging

To avoid averaging across edges, assume bimodal greyvalue distribution and select average value of modality with largest population.

1. Determine $\bar{g}_D = \frac{1}{|D|} \sum_{g_{mn} \in D} g_{mn}$

2. $A = \{g_k \text{ with } g_k \geq \bar{g}\}$ $\quad B = \{g_k \text{ with } g_k < \bar{g}\}$

3. $g' = \begin{cases} \frac{1}{|A|} \sum_{g_k \in A} g_k & \text{if } |A| \geq |B| \\ \frac{1}{|B|} \sum_{g_k \in B} g_k & \text{otherwise} \end{cases}$

Example:

B

11	14	15
13	12	25
15	19	26

A

$\bar{g} = 16{,}7$ ⇒ **A, B** ⇒ $g' = 13$

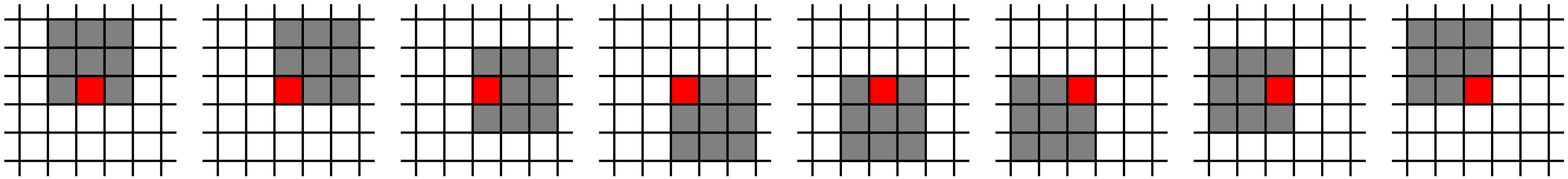
Averaging with Rotating Mask

Replace center pixel by average over pixels from the most homogeneous subset taken from the neighbourhood of center pixel.

Measure for (lack of) homogeneity is dispersion σ^2 (= empirical variance) of the greyvalues of a region D:

$$\bar{g}_{ij} = \frac{1}{|D|} \sum_{g_{mn} \in D} g_{mn} \qquad \sigma_{ij}^2 = \frac{1}{|D|} \sum_{g_{mn} \in D} (g_{mn} - \bar{g}_{ij})^2$$

Possible rotated masks in 5 x 5 neighbourhood of center pixel:



<u>Algorithm</u>:

1. Consider each pixel g_{ij}
2. Calculate dispersion in mask for all rotations
3. Choose mask with minimum dispersion
4. Assign average greyvalue of chosen mask to g_{ij}

Median Filter

Median of a distribution P(x): x_m such that $P(x < x_m) = 1/2$

Median Filter:

$$\hat{g}_{ij} = \max a \text{ with } g_k \in D \text{ and } |\{g_k < a\}| < \frac{|D|}{2}$$

1. Sort pixels in D according to greyvalue
2. Choose greyvalue in middle position

Example:

11	14	15
13	12	25
15	19	26

11
12
13
14
15
15
19
25
26

greyvalue of center pixel of region is set to 15

Median Filter reduces influence of outliers in either direction!

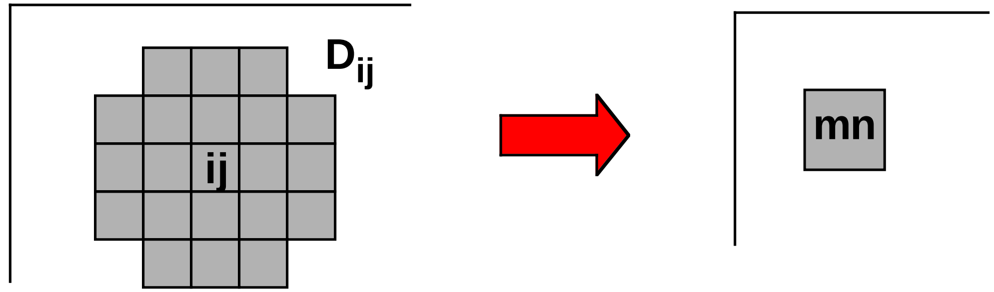
Local Neighbourhood Operations

Many useful image transformations may be defined as an instance of a local neighbourhood operation:

Generate a new image with pixels $\hat{g}_{mn}$ by applying operator f to all pixels g_{ij} of an image

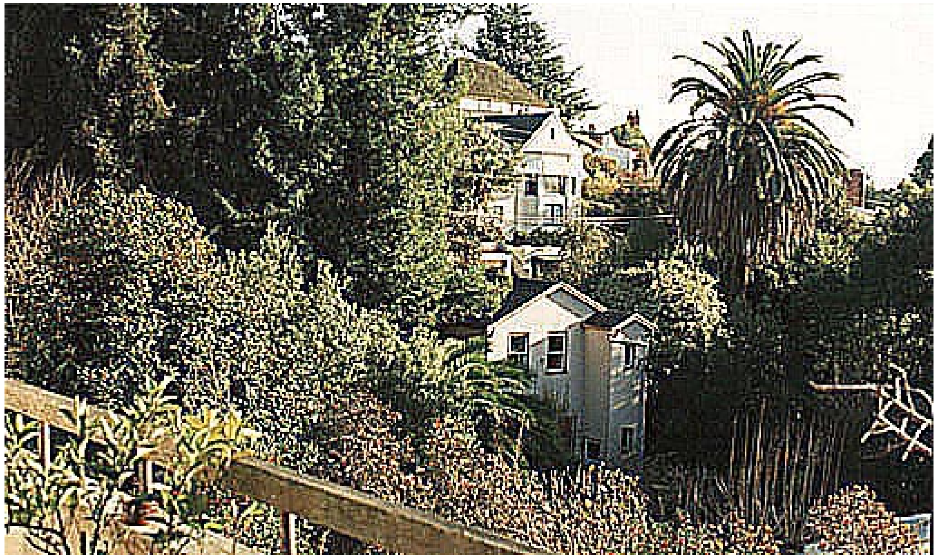
$$\hat{g}_{mn} = f(g_1, g_2, \ldots, g_K) \qquad g_1, g_2, \ldots, g_K \in D_{ij}$$

example of neighbour-hood



Pixel indices i, j may be incremented by steps larger than 1 to obtain reduced new image.

Example of Sharpening



intensity sharpening with 3 x 3 mask

-1	-1	-1
-1	9	-1
-1	-1	-1

"unsharp masking" = subtraction of blurred image

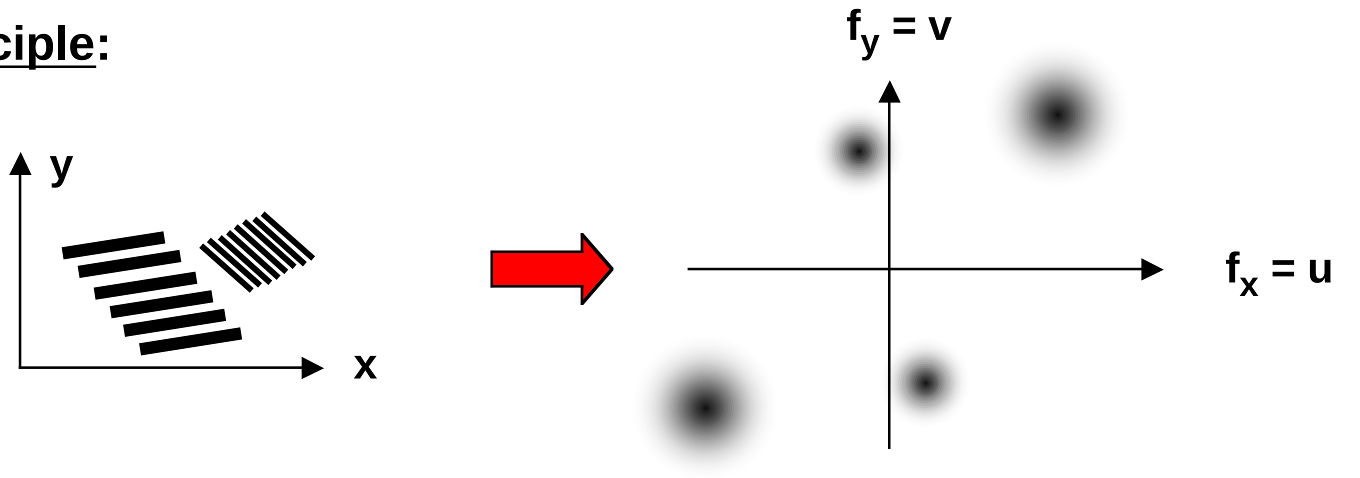
$$\hat{g}_{ij} = g_{ij} - \frac{1}{|D|} \sum_{g_{mn} \in D} g_{mn}$$

Spectral Image Properties

An image function may be considered a sum of spatially sinusoidal components of different frequencies.

The frequency spectrum indicates the magnitudes of the spatial frequencies contained in an image.

<u>Principle</u>:

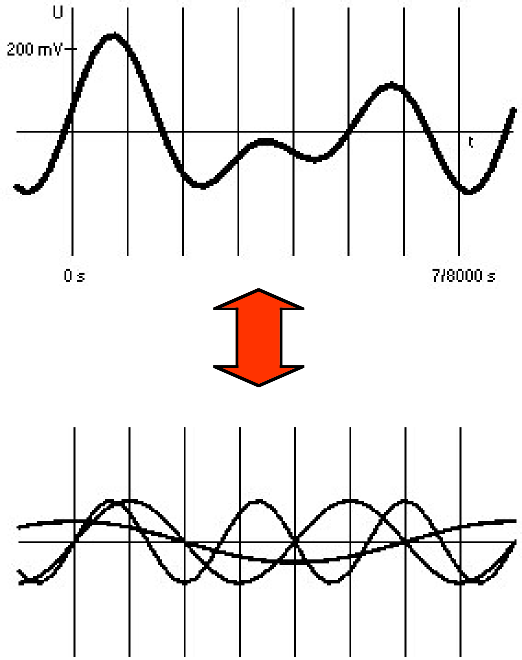


Important qualitative properties of spectral information:

- **spectral information is independent of image locations**
- **sharp edges give rise to high frequencies**
- **noise (= disturbances of image signal) is often high-frequency**

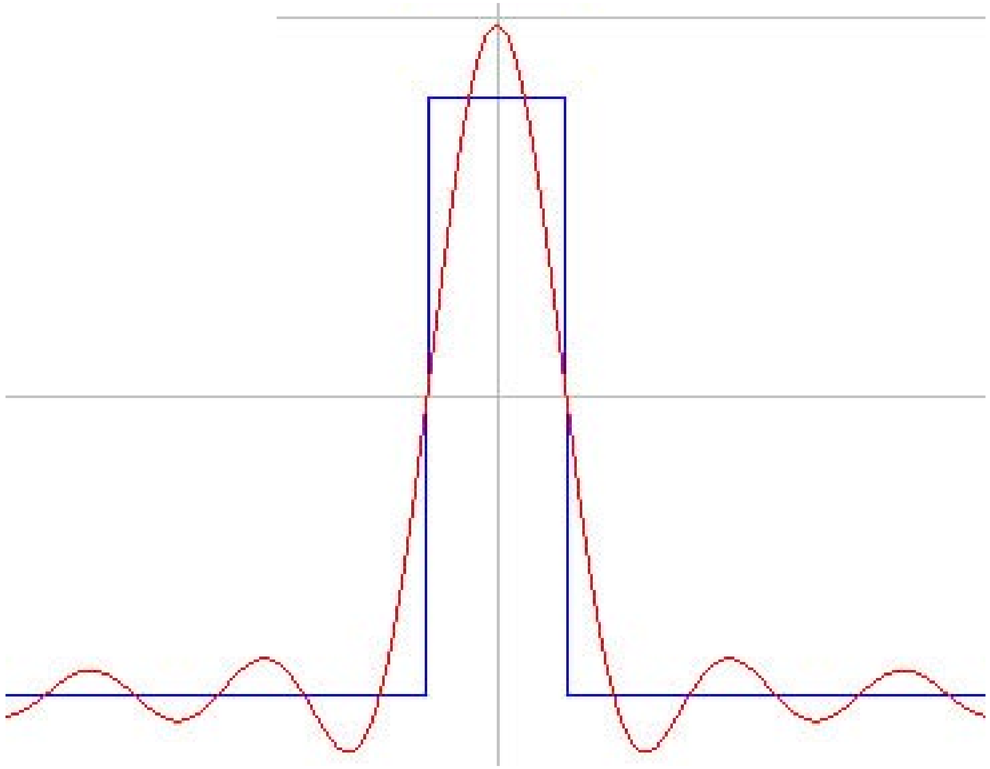
Illustration of
1-D Fourier Series Expansion

original waveform



sinusoidal components add up to original waveform

approximation of a rectangular pulse with 1 ... 5 sinusoidal components



Online demonstration of Fourier Series approximations at http://www.jhu.edu/~signals/fourier2/

Discrete Fourier Transform (DFT)

Computes image representation as a sum of sinusoidals.

Discrete Fourier Transform:

$$G_{uv} = \frac{1}{MN} \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} g_{mn} e^{-2\pi i (\frac{mu}{M} + \frac{nv}{N})}$$

for u = 0 ... M-1, v = 0 ... N-1

Inverse Discrete Fourier Transform:

$$g_{mn} = \sum_{u=0}^{M-1} \sum_{v=0}^{N-1} G_{uv} e^{2\pi i (\frac{mu}{M} + \frac{nv}{N})}$$

for m = 0 ... M-1, n = 0 ... N-1

Notation for computing the Fourier Transform:

$G_{uv} = \mathbf{F}\{ g_{mn} \}$

$g_{mn} = \mathbf{F}^{-1}\{ G_{uv} \}$

Transform is based on periodicity assumption

=> periodic continuation may cause boundary effects

Basic Properties of DFT

- **Linearity:** $\mathcal{F}\{ a\, g_{mn} + b\, g_{mn}\} = a\,\mathcal{F}\{ g_{mn} \} + b\,\mathcal{F}\{ g_{mn} \}$
- **Symmetry:** $G_{-u,-v} = G_{uv}$ for real g_{mn} (such as images)

In general, the Fourier transform is a complex function with a real and an imaginary part:

$G_{uv} = R_{uv} + i\, I_{uv}$

Euler´s formula:
$r\, e^{iz} = r \cos(z) + r\, i \sin(z)$

$$|G_{uv}| = \sqrt{R_{uv}^2 + I_{uv}^2}$$ **frequency spectrum or amplitude spectrum**

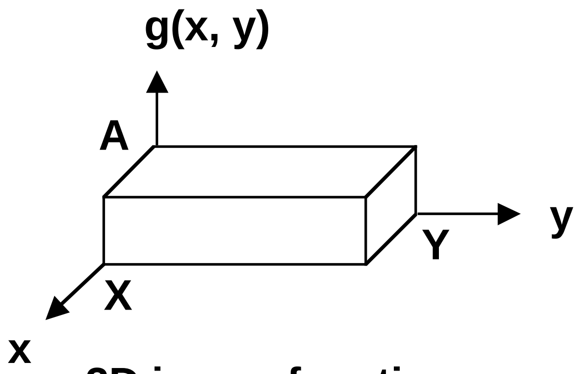
$$P_{uv} = |G_{uv}|^2 = R_{uv}^2 + I_{uv}^2$$ **power spectrum or spectral density**

$$\Phi_{uv} = \tan^{-1}\left(\frac{I_{uv}}{R_{uv}}\right)$$ **phase spectrum**

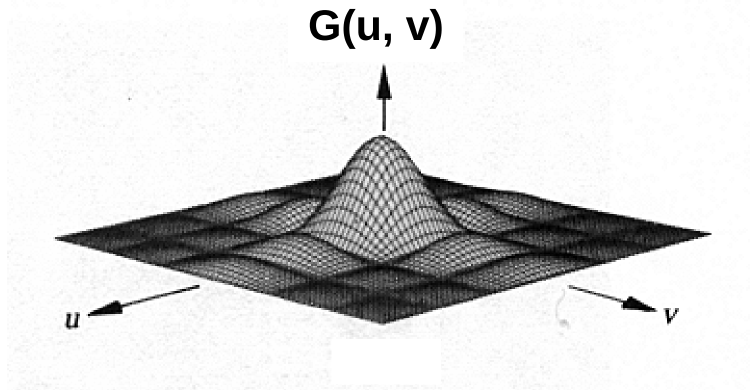
Recommended reading:

Gonzalez/Wintz
Digital Image Processing
Addison Wesley 87

Illustrative Example of Fourier Transform

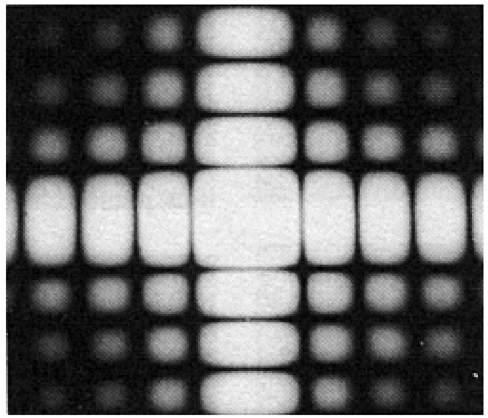


2D image function



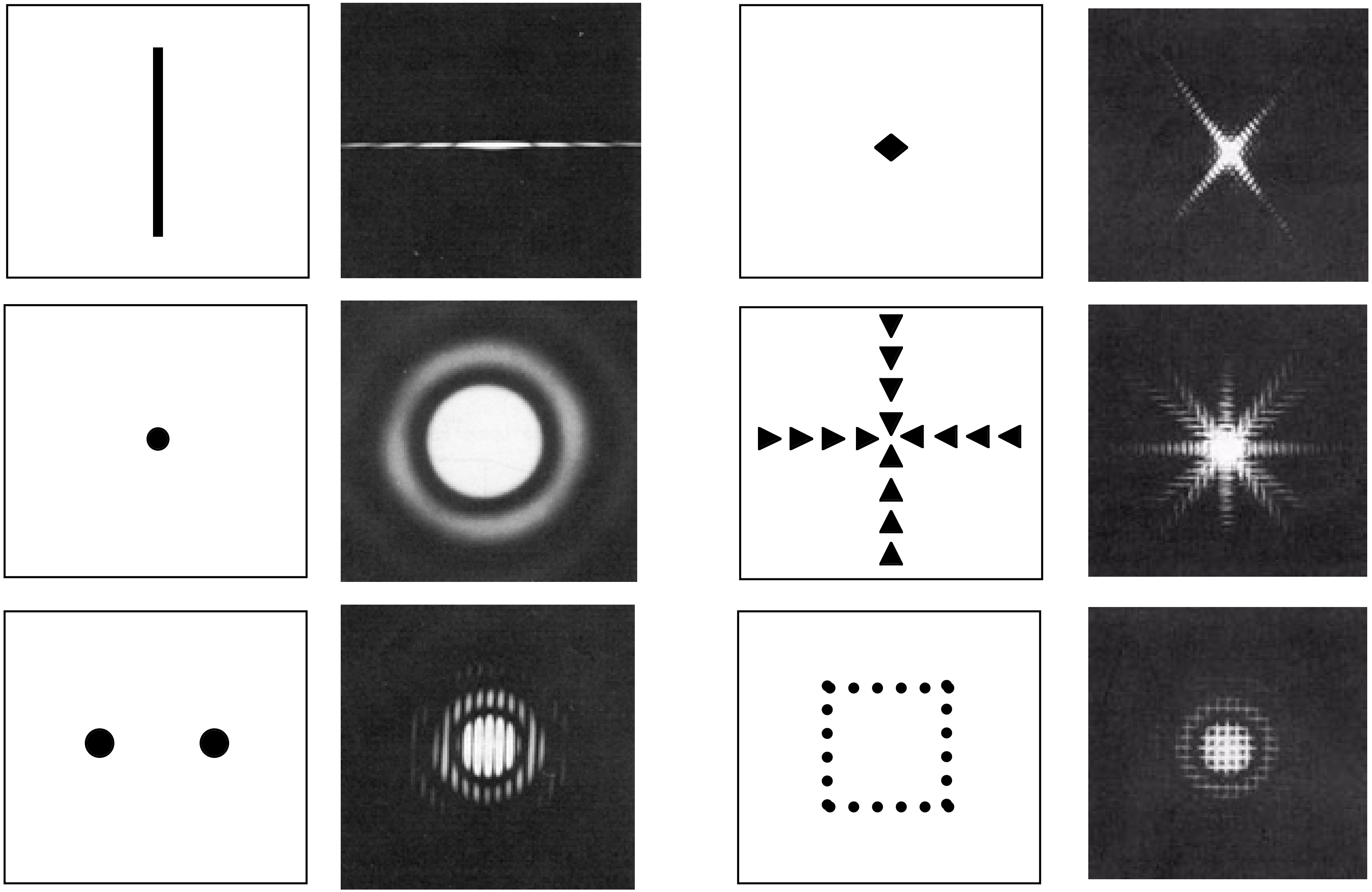
frequency spectrum

Note that large spectral amplitudes occur in directions vertical to prominent edges of the image function



frequency spectrum as an intensity function

Examples of Fourier Transform Pairs



Fast Fourier Transform (FFT)

Ordinary DFT needs ~$(MN)^2$ operations for an M x N image.

Example: M = N = 512, 10^{-9} sec/operation => 64 sec

FFT is based on recursive decomposition of g_{mn} into subsequences.
=> multiple use of partial results => ~MN $\log_2$(MN) operations
Same example needs only 0.0046 sec

Decomposition principle for 1D Fourier transform:

$$G_r = \frac{1}{N}\sum_{n=0}^{N-1} g_n e^{-2\pi i r \frac{n}{N}}$$

$$\{ g_n \} = \begin{cases} \{ g_n^{(1)} \} = \{ g_{2n} \} \\ \{ g_n^{(2)} \} = \{ g_{2n+1} \} \end{cases} \quad n = 0 \;..\; N/2-1$$

$$G_r = \frac{1}{N}\sum_{n=0}^{\frac{N}{2}-1} \left\{ g_n^{(1)} e^{-2\pi i r \frac{2n}{N}} + g_n^{(2)} e^{-2\pi i r \frac{(2n+1)}{N}} \right\} \quad r = 0 \;...\; N-1$$

$$G_r = G_r^{(1)} + e^{-2\pi i \frac{r}{N}} G_r^{(2)}$$

$$G_{r+N/2} = G_r^{(1)} - e^{-2\pi i \frac{r}{N}} G_r^{(2)} \quad r = 0 \;...\; N/2-1$$

All G_r may be computed by $2(N/2)^2$ instead of $(N)^2$ operations!

Convolution

Convolution is an important operation for describing and analyzing linear operations, e.g. filtering.

Definition of 2D convolution for continuous signals:

$$g(x,y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(r,s)\, h(x-r, y-s)\, dr\, ds = f(x,y) * h(x,y)$$

Convolution in the spatial domain is dual to multiplication in the frequency domain:

$$\mathbf{F}\{ f(x, y) * h(x, y) \} = F(u, v)\, H(u, v)$$

$$\mathbf{F}\{ f(x, y)\, h(x, y) \} = F(u, v) * H(u, v)$$

H can be interpreted as attenuating or amplifying the frequencies of F.

=> Convolution describes <u>filtering</u> in the spatial domain.

Filtering in the Frequency Domain

A filter transforms a signal by modifying its spectrum.

$$G(u, v) = F(u, v)\,H(u, v)$$

H is the frequency transfer function of the filter.

- **low-pass filter** — ***low frequencies pass, high frequencies are attenuated or removed***
- **high-pass filter** — ***high frequencies pass, low frequencies are attenuated or removed***
- **band-pass filter** — ***frequencies within a frequency band pass, other frequencies below or above are attenuated or removed***

Often (but not always) the noise part of an image is high-frequency and the signal part is low-frequency. Low-pass filtering then improves the signal-to-noise ratio.

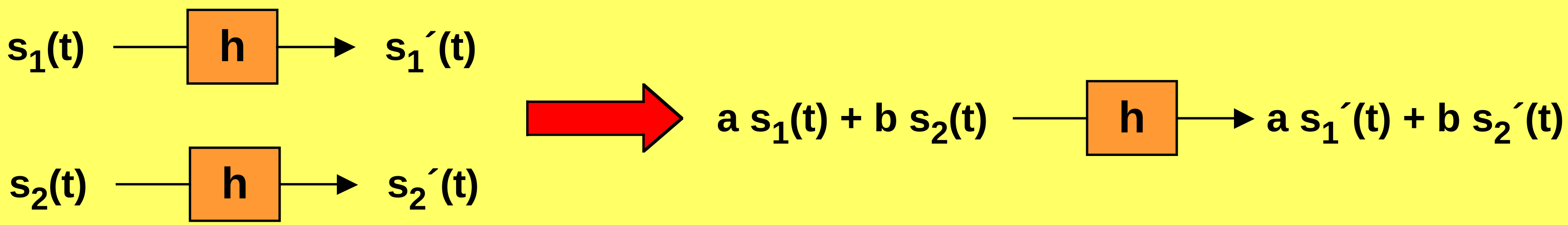
Filtering in the Spatial Domain

Filtering in the spatial domain is described by convolution.

$$g(x,y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(r,s)\, h(x-r, y-s)\, dr\, ds = f(x,y) * h(x,y)$$

Commonly used description for the effect of technical components in linear signal theory:

$$s'(t) = \int_{-\infty}^{+\infty} h(r)\, s(t-r)\, dr$$



An impulse δ as input generates the filter function h(x, y) as output:

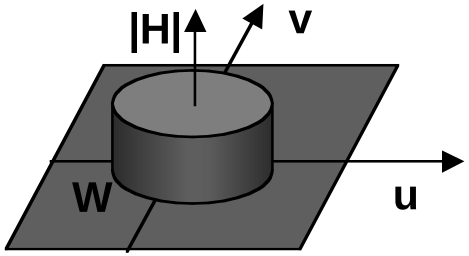
$$h(x,y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(r,s)\, \delta(x-r, y-s)\, dr\, ds = h(x,y) * \delta(x,y)$$

h(x, y) is often called "impulse response"

Low-pass Filters

Ideal low-pass filter

All frequencies above W are removed



$$|H(u, v)| = \begin{cases} 1 & \text{for } \sqrt{u^2 + v^2} \le W \\ 0 & \text{otherwise} \end{cases}$$

Note that the filter function h(x, y) is rotation symmetric and
$h(r) \sim \sin 2\pi Wr \,/\, (2\pi Wr)$ with $r^2 = x^2 + y^2$
=> impuls-shaped input structures may produce ring-like structures as output

Gaussian filter

A Gaussian filter has an optimally smooth boundary, both in the frequency and the spatial domain. It is important for several advanced image analysis methods, e.g. generating multiscale images.

$$H(u,v) = e^{-\frac{1}{2}(u^2+v^2)\sigma^2} \qquad h(x,y) = \frac{1}{2\pi\sigma^2} e^{-\frac{1}{2}\frac{x^2+y^2}{\sigma^2}}$$

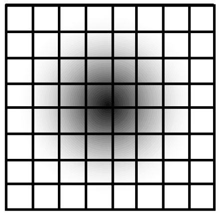
Discrete Filters

For periodic discrete 2D signals (e.g. discrete images), the convolution operator which describes filtering is

$$g_{ij} = \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} f_{mn} h_{i-m,j-n}$$

Each pixel g_{ij} of the filtered image is the sum of the products of the original image with the mirror filter $h_{-m,-n}$ placed at location ij.

Example:



$h_{mn} = h_{-m,-n}$ is a bell-shaped function

The filtering effect is a smoothing operation by weighted local averaging.

The choice of weights of a local filter - the convolution mask - may influence the properties of the output image in important ways, e.g. with regard to remaining noise, blurred edges, artificial structures, preserved or discarded information.

Matrix Notation for Discrete Filters

The convolution operation $$g_{ij} = \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} f_{mn} h_{i-m,j-n}$$

may be expressed as matrix multiplication $\underline{g} = H\,\underline{f}$.

Vectors $\underline{g}$ and $\underline{f}$ are obtained by stacking rows (or columns) onto each other:

$$\underline{g}^T = [g_{00}\ g_{01}\ \ldots\ g_{0\,N\text{-}1}\ g_{10}\ g_{11}\ \ldots\ g_{1N\text{-}1}\ \ldots\ g_{M\text{-}1\,0}\ g_{M\text{-}1\,1}\ \ldots\ g_{M\text{-}1\,N\text{-}1}]$$
$$\underline{f}^T = [f_{00}\ f_{01}\ \ldots\ f_{0\,N\text{-}1}\ f_{10}\ f_{11}\ \ldots\ f_{1\,N\text{-}1}\ \ldots\ f_{M\text{-}1\,0}\ f_{M\text{-}1\,1}\ \ldots\ f_{M\text{-}1\,N\text{-}1}]$$

The filter matrix H is obtained by constructing a matrix H_j for each row j of h_{ij}:

$$H_j = \begin{bmatrix} h_{j\,0} & h_{j\,N\text{-}1} & h_{j\,N\text{-}2} & \ldots & h_{j\,1} \\ h_{j\,1} & h_{j\,0} & h_{j\,N\text{-}1} & \ldots & h_{j\,2} \\ \vdots & & & & \\ h_{j\,N\text{-}1} & h_{j\,N\text{-}2} & h_{1\,N\text{-}3} & \ldots & h_{j\,0} \end{bmatrix}$$

$$H = \begin{bmatrix} H_0 & H_{M\text{-}1} & H_{M\text{-}2} & \ldots & H_1 \\ H_1 & H_0 & H_{M\text{-}1} & \ldots & H_2 \\ \vdots & & & & \\ H_{M\text{-}1} & H_{M\text{-}2} & H_{M\text{-}3} & \ldots & H_0 \end{bmatrix}$$

Avoiding Wrap-around Errors

Wrap-around errors result from filter responses due to the periodic continuation of image and filter.

To avoid wrap-around errors, image and filter have to be extended by zeros.

To avoid wrap-around error:

$$M \geq A + C - 1$$
$$N \geq B + D - 1$$

A x B	**original image size**
C x D	**original filter size**
M x N	**extended image and filter size**

Example:

Convolution Using the FFT

Convolution in the spatial domain may be performed more efficiently using the FFT.

$$g'_{ij} = \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} g_{mn} h_{i-m,j-n} \qquad (MN)^2 \text{ operations needed}$$

Using the FFT and filtering in the frequency domain:

$$g_{mn} \xrightarrow{\text{FFT}} G_{uv} \xrightarrow{H_{uv}} G'_{uv} \xrightarrow{\text{FFT}^{-1}} g'_{mn}$$

MN log(MN)	MN	MN log(MN)	# of operations

Example with M = N = 512:

- **straight convolution needs ~ 10^{10} operations**
- **convolution using the FFT needs ~10^{7} operations**

Convolution and Correlation

The crosscorrelation function of 2 stationary stochastic processes f and h is:

$$g(x,y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(r,s)\, h(r-x, s-y)\, dr\, ds = f(x,y) \circ h(x,y) = f(x,y) * h(-x,-y)$$

Compare with convolution: filter function is not mirrored!

Correlation using Fourier Transform:

$\mathbf{F}\{ f(x, y) \circ h(x, y) \} = F^*(u, v)\, H(u, v)$

$\mathbf{F}\{ f^*(x, y)\, h(x, y) \} = F(u, v) \circ H(u, v)$

F*, f* are complex conjugates

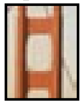
Correlation is particularly important for matching problems, e.g. matching an image with a template.

Correlation may be computed more efficiently by using the FFT.

Correlation and Matching

Matching a template with an image:

image **template**



- **find degree of match for all locations of template**
- **find location of best match**

For (periodic) discrete images, crosscorrelation at (i, j) is

$$c_{ij} = \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} f_{mn} h_{m-i,n-j}$$

Compare with Euclidean distance between f and h at location (i, j):

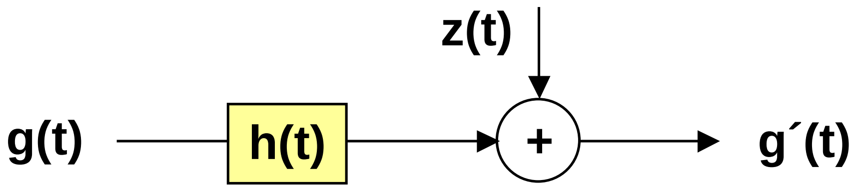
$$d_{ij} = \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} \left(f_{mn} - h_{m-i,n-j} \right)^2 =$$

$$\sum_{m=0}^{M-1} \sum_{n=0}^{N-1} \left(f_{mn} \right)^2 - 2 \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} f_{mn} h_{m-i,n-j} + \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} \left(h_{m-i,n-j} \right)^2$$

Since image energy and template energy are constant, correlation measures distance

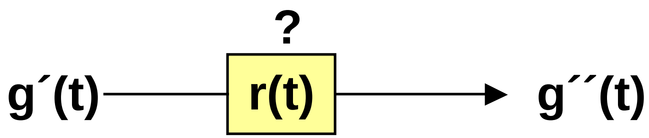
Principle of Image Restoration

Typical degradation model of a continuous 1-dimensional signal:



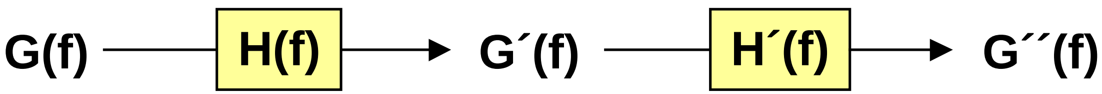
g(t)	original signal
h(t)	degrading filter
z(t)	additive noise
g´(t)	degraded signal

How can one process g´(t) to obtain a g´´(t) which best approximates g(t)?



r(t)	restoring filter
g´´(t)	restored signal

Note that a perfect restoration g´´(t) = g(t) may not be possible even if z(t) ≡0.



The ideal restoring filter H´(f) = 1/H(f) may not exist because of zeros of H(f).

Image Restoration by Minimizing the MSE

Degradation in matrix notation: $\underline{g}' = H\,\underline{g} + \underline{z}$

Restored signal $\underline{g}''$ must minimize the mean square error $J(\underline{g}'')$ of the remaining difference:

$$\min \| \underline{g}' - H\underline{g}'' \|^2$$

$$\delta J(\underline{g}'')/\delta \underline{g}'' = 0 = -2H^T(\underline{g}' - H\underline{g}'')$$

$$\underline{g}'' = \underline{(H^T H)^{-1} H^T}\,\underline{g}'$$

pseudoinverse of H

If M = N and hence H is a square matrix, and if H^{-1} exists, we can simplify:

$$\underline{g}'' = H^{-1}\underline{g}'$$

The matrix H^{-1} gives a perfect restoration if $\underline{z} \equiv 0$.