Perspective Projection Transformation

Where does a point of a scene appear in an image?

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} \xrightarrow{?} \begin{bmatrix} x_p'' \\ y_p'' \end{bmatrix}$$

Transformation in 3 steps:

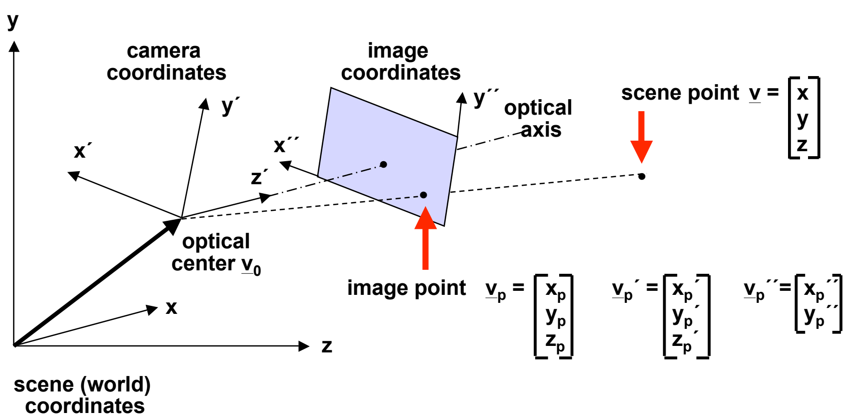
1. **scene coordinates => camera coordinates**
2. **projection of camera coordinates into image plane**
3. **camera coordinates => image coordinates**

Perspective projection equations are essential for Computer Graphics. For Image Understanding we will need the inverse: What are possible scene coordinates of a point visible in the image? This will follow later.

1

Perspective Projection in Independent Coordinate Systems

It is often useful to describe real-world points, camera geometry and image points in separate coordinate systems. The formal description of projection involves transformations between these coordinate systems.



2

3D Coordinate Transformation (1)

The new coordinate system is specified by a translation and rotation with respect to the old coordinate system:

$\underline{v}' = R(\underline{v} - \underline{v}_0)$ $\underline{v}_0$ is displacement vector; R is rotation matrix

Note that these matrices describe coo transforms for positive rotations of the coo system.

R may be decomposed into 3 rotations about the coordinate axes:
$R = R_x R_y R_z$

$$R_x = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & \sin\alpha \\ 0 & -\sin\alpha & \cos\alpha \end{bmatrix}$$

If rotations are performed in the above order:
1) γ = rotation angle about z-axis
2) β = rotation angle about (new) y-axis
3) α = rotation angle about (new) x-axis

("tilt angle", "pan angle", and "nick angle" for the camera coordinate assignment shown before)

$$R_y = \begin{bmatrix} \cos\beta & 0 & -\sin\beta \\ 0 & 1 & 0 \\ \sin\beta & 0 & \cos\beta \end{bmatrix}$$

$$R_z = \begin{bmatrix} \cos\gamma & \sin\gamma & 0 \\ -\sin\gamma & \cos\gamma & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

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3D Coordinate Transformation (2)

By multiplying the 3 matrices R_x, R_y and R_z, one gets

$$R = \begin{bmatrix} \cos\beta\cos\gamma & \cos\beta\sin\gamma & -\sin\beta \\ \sin\alpha\sin\beta\cos\gamma - \cos\alpha\sin\gamma & \sin\alpha\sin\beta\sin\gamma + \cos\alpha\cos\gamma & \sin\alpha\cos\beta \\ \cos\alpha\sin\beta\cos\gamma + \sin\alpha\sin\gamma & \cos\alpha\sin\beta\sin\gamma - \sin\alpha\cos\gamma & \cos\alpha\cos\beta \end{bmatrix}$$

For formula manipulations, one tries to avoid the trigonometric functions and takes

$$R = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix}$$

Note that the coefficients of R are constrained:
A rotation matrix is orthonormal:

$R R^T = I$ (unit matrix)

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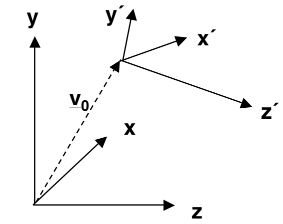
Example for Coordinate Transformation



camera coo system:

- displacement by $\underline{v}_0$
- rotation by pan angle $\beta = -30^0$
- rotation by nick angle $\alpha = 45^0$

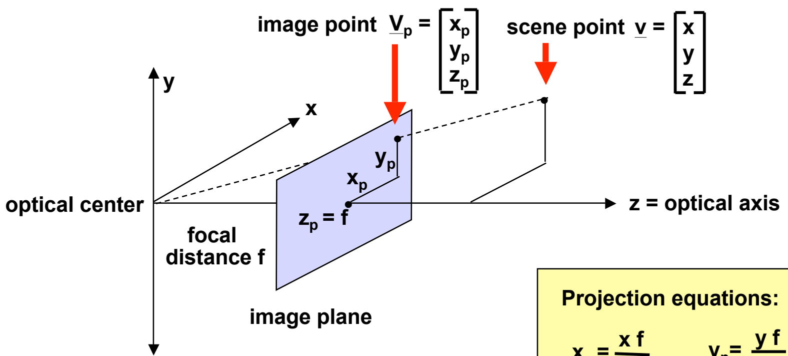
$\underline{v}' = R\,(\underline{v} - \underline{v}_0)$ with R = Rx Ry

$$R_x = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2}\sqrt{2} & \frac{1}{2}\sqrt{2} \\ 0 & -\frac{1}{2}\sqrt{2} & \frac{1}{2}\sqrt{2} \end{bmatrix} \qquad R_y = \begin{bmatrix} \frac{1}{2}\sqrt{3} & 0 & \frac{1}{2} \\ 0 & 1 & 0 \\ -\frac{1}{2} & 0 & \frac{1}{2}\sqrt{3} \end{bmatrix}$$

Perspective Projection Geometry

Projective geometry relates the coordinates of a point in a scene to the coordinates of its projection onto an image plane.

Perspective projection is an adequate model for most cameras.



Projection equations:

$x_p = \frac{x\,f}{z}$ $\qquad y_p = \frac{y\,f}{z}$

Perspective and Orthographic Projection

Within the camera coordinate system the perspective projection of a scene point onto the image plane is described by

$x_p' = \frac{x'f}{z'}$ $y_p' = \frac{y'f}{z'}$ $z_p' = f$ (f = focal distance)

- nonlinear transformation
- loss of information

If all objects are far away (large z'), f/z' is approximately constant

=> orthographic projection

$x_p' = s\,x'$ $y_p' = s\,y'$ (s = scaling factor)

Orthographic projection can be viewed as projection with parallel rays + scaling

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From Camera Coordinates to Image Coordinates

Transform may be necessary because

- optical axis may not penetrate image plane at origin of desired coordinate system
- transition to discrete coordinates may require scaling

$x_p'' = (x_p' - x_{p0}')\,a$ a, b scaling parameters

$y_p'' = (y_p' - y_{p0}')\,b$ x_{p0}', y_{p0}' origin of image coordinate system

Example:

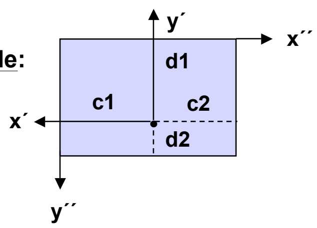

Image boundaries in camera coordinates:
$x'_{max} = c1$ $x'_{min} = c2$
$y'_{max} = d1$ $y'_{min} = d2$

Discrete image coordinates:
$x'' = 0 .. 511$ $y'' = 0 .. 575$

Transformation parameters:
$x_{p0}' = c1$ $y_{p0}' = d1$ $a = 512 / (c2 - c1)$ $b = 576 / (d2 - d1)$

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Complete Perspective Projection Equation

We combine the 3 transformation steps:

1. scene coordinates => camera coordinates
2. projection of camera coordinates into image plane
3. camera coordinates => image coordinates

$x_p'' = \{ f/z' [\cos\beta \cos\gamma \ (x - x_0) + \cos\beta \sin\gamma \ (y - y_0) + \sin\beta \ (z - z_0)] - x_{p0} \} \ a$

$y_p'' = \{ f/z' [(-\sin\alpha \sin\beta \cos\gamma - \cos\alpha \sin\gamma) \ (x - x_0) +$
$(-\sin\alpha \sin\beta \sin\gamma + \cos\alpha \cos\gamma) \ (y - y_0) +$
$\sin\alpha \cos\beta \ (z - z_0)] - y_{p0} \} \ b$

with $z' = (-\cos\alpha \sin\beta \cos\gamma + \sin\alpha \sin\gamma)(x - x_0) +$
$(-\cos\alpha \sin\beta \sin\gamma - \sin\alpha \cos\gamma)(y - y_0) +$
$\cos\alpha \cos\beta (z - z_0)$

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Homogeneous Coordinates (1)

4D notation for 3D coordinates which allows to express nonlinear 3D transformations as linear 4D transformations.

Normal: $\underline{v}' = R \ (\underline{v} - \underline{v}_0)$

Homogeneous coordinates: $\underline{\mathit{v}}' = \mathit{A} \ \underline{\mathit{v}}$

(note italics for homogeneous coordinates)

$$A = R\,T = \begin{bmatrix} r_{11} & r_{12} & r_{13} & 0 \\ r_{21} & r_{22} & r_{23} & 0 \\ r_{31} & r_{32} & r_{33} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & -x_0 \\ 0 & 1 & 0 & -y_0 \\ 0 & 0 & 1 & -z_0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Transition to homogeneous coordinates:

$\underline{v}^T = [x \ y \ z]$ => $\underline{\mathit{v}}^T = [wx \ wy \ wz \ w]$ $\quad w \neq 0$ is arbitrary constant

Return to normal coordinates:

1. Divide components 1- 3 by 4th component
2. Omit 4th component

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Homogeneous Coordinates (2)

Perspective projection in homogeneous coordinates:

$$\underline{v}_p{}' = P\,\underline{v}' \quad \text{with } P = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1/f & 0 \end{bmatrix} \quad \text{and } \underline{v}' = \begin{bmatrix} wx \\ wy \\ wz \\ w \end{bmatrix} \quad \text{gives } \underline{v}_p{}' = \begin{bmatrix} wx \\ wy \\ wz \\ wz/f \end{bmatrix}$$

Returning to normal coordinates gives $\underline{v}_p{}' = \begin{bmatrix} xf/z \\ yf/z \\ f \end{bmatrix}$

compare with earlier slide

Transformation from camera into image coordinates:

$$\underline{v}_p{}'' = B\,\underline{v}_p{}' \quad \text{with } B = \begin{bmatrix} a & 0 & 0 & -x_0 a \\ 0 & b & 0 & -y_0 b \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad \text{and } \underline{v}_p{}' = \begin{bmatrix} wx_p \\ wy_p \\ 0 \\ w \end{bmatrix} \quad \text{gives } \underline{v}_p{}'' = \begin{bmatrix} wa(x_p-x_0) \\ wb(y_p-y_0) \\ 0 \\ w \end{bmatrix}$$

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Homogeneous Coordinates (3)

Perspective projection can be completely described in terms of a linear transformation in homogeneous coordinates:

$$\underline{v}_p{}'' = B\,P\,R\,T\,\underline{v}$$

B P R T may be combined into a single 4 x 4 matrix *C* :

$$\underline{v}_p{}'' = C\,\underline{v}$$

In the literature the parameters of these equations may vary because of different choices of coordinate systems, different order of translation and rotation, different camera models, etc.

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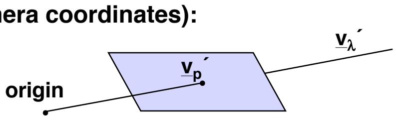
Inverse Perspective Equations

Which points in a scene correspond to a point in the image?

$$\begin{bmatrix} x_p'' \\ y_p'' \end{bmatrix} \xrightarrow{?} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

Each image point defines a projection ray as the locus of possible scene points (for simplicity in camera coordinates):



$\underline{v}_p{}' \;\Rightarrow\; \underline{v}_\lambda{}' = \lambda\, \underline{v}_p{}'$

$\underline{v} = \underline{v}_0 + R^T \lambda\, \underline{v}_p{}'$

3 equations with the 4 unknowns x, y, z, λ and camera parameters R and $\underline{v}_0$

Applications of inverse perspective mapping for e.g.

- distance measurements
- binocular stereo
- camera calibration
- motion stereo

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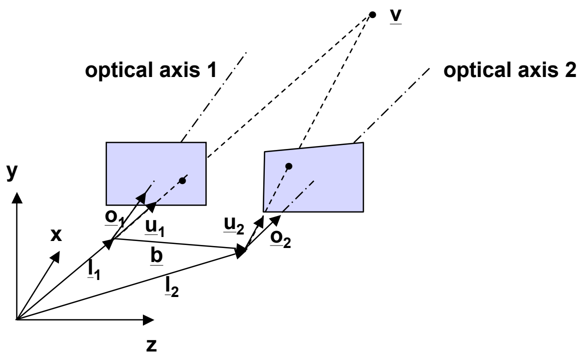
Binocular Stereo (1)



$\underline{l}_1$, $\underline{l}_2$	camera positions (optical center)
$\underline{b}$	stereo base
$\underline{o}_1$, $\underline{o}_2$	camera orientations (unit vectors)
f_1, f_2	focal distances
$\underline{v}$	scene point
$\underline{u}_1$, $\underline{u}_2$	projection rays of scene point (unit vectors)

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Binocular Stereo (2)

Determine distance to $\underline{v}$ by measuring $\underline{u}_1$ and $\underline{u}_2$

Formally: $\alpha \underline{u}_1 = \underline{b} + \beta \underline{u}_2 \Rightarrow \underline{v} = \alpha \underline{u}_1 + \underline{l}_1$

α and β are overconstrained by the vector equation. In practice, measurements are inexact, no exact solution exists (rays do not intersect).

Better approach: Solve for the point of closest approximation of both rays:

$$\underline{v} = \frac{\alpha_0 \underline{u}_1 + (\underline{b} + \beta_0 \underline{u}_2)}{2} + \underline{l}_1 \quad \Rightarrow \quad \text{minimize} \quad \| \alpha \underline{u}_1 - (\underline{b} + \beta \underline{u}_2) \|^2$$

Solution:

$$\alpha_0 = \frac{\underline{u}_1^T \underline{b} - (\underline{u}_1^T \underline{u}_2)(\underline{u}_2^T \underline{b})}{1 - (\underline{u}_1^T \underline{u}_2)^2}$$

$$\beta_0 = \frac{(\underline{u}_1^T \underline{u}_2)(\underline{u}_1^T \underline{b}) - (\underline{u}_2^T \underline{b})}{1 - (\underline{u}_1^T \underline{u}_2)^2}$$

Distance in Digital Images

Intuitive concepts of continuous images do not always carry over to digital images.

Several methods for measuring distance between pixels:

Eucledian distance

$D_E((i,j),(h,k)) = \sqrt{(i-h)^2 + (j-k)^2}$ — costly computation of square root, can be avoided for distance comparisons

City-block distance

$D_4((i, j)(h, k)) = |i - h| + |j - k|$ — number of horizontal and vertical steps in a rectangular grid

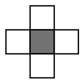
Chessboard distance

$D_8((i, j)(h, k)) = \max \{ |i - h|, |j - k| \}$ — number of steps in a rectangular grid if diagonal steps are allowed (number of moves of a king on a chessboard)

Connectivity in Digital Images

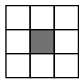
Connectivity is an important property of subsets of pixels. It is based on adjacency (or neighbourhood):

Pixels are 4-neighbours if their distance is $D_4 = 1$



all 4-neighbours of center pixel

Pixels are 8-neighbours if their distance is $D_8 = 1$



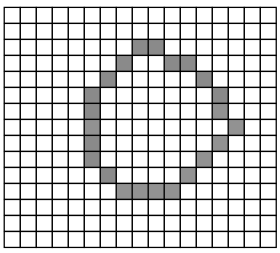
all 8-neighbours of center pixel

A path from pixel P to pixel Q is a sequence of pixels beginning at Q and ending at P, where consecutive pixels are neighbours.

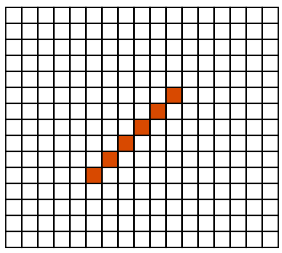
In a set of pixels, two pixels P and Q are connected, if there is a path between P and Q with pixels belonging to the set.

A region is a set of pixels where each pair of pixels is connected.

Closed Curve Paradoxon



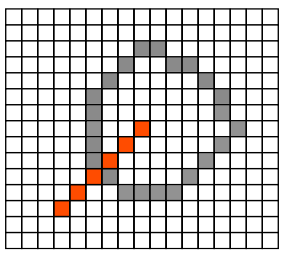
line 1



line 2

solid lines if 8-neighbourhood is used

a similar paradoxon arises if 4-neighbourhoods are used



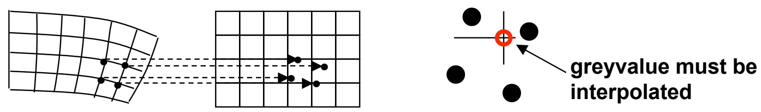
line 2 does not intersect line 1 although it crosses from the outside to the inside

Geometric Transformations

Various applications:

- change of view point
- elimination of geometric distortions from image capturing
- registration of corresponding images
- artificial distortions, Computer Graphics applications

Step 1: Determine mapping $\underline{T}$(x, y) from old to new coordinate system
Step 2: Compute new coordinates (x´, y´) for (x, y)
Step 3: Interpolate greyvalues at grid positions from greyvalues at transformed positions



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Polynomial Coordinate Transformations

General format of transformation:

$$x' = \sum_{r=0}^{m} \sum_{k=0}^{m-r} a_{rk} x^r y^k$$

$$y' = \sum_{r=0}^{m} \sum_{k=0}^{m-r} b_{rk} x^r y^k$$

- Assume polynomial mapping between (x, y) and (x´, y´) of degree m
- Determine corresponding points
- a) Solve linear equations for a_{rk}, b_{rk} (r, k = 1 ... m)
  b) Minimize mean square error (MSE) for point correspondences

Approximation by biquadratic transformation:

$x' = a_{00} + a_{10}x + a_{01}y + a_{11}xy + a_{20}x^2 + a_{02}y^2$
$y' = b_{00} + b_{10}x + b_{01}y + b_{11}xy + b_{20}x^2 + b_{02}y^2$

at least 6 corresponding pairs needed

Approximation by affine transformation:

$x' = a_{00} + a_{10}x + a_{01}y$
$y' = b_{00} + b_{10}x + b_{01}y$

at least 3 corresponding pairs needed

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Translation, Rotation, Scaling, Skewing

Translation by vector $\underline{t}$:

$\underline{v}' = \underline{v} + \underline{t}$ with $\underline{v}' = \begin{bmatrix} x' \\ y' \end{bmatrix}$ $\underline{v} = \begin{bmatrix} x \\ y \end{bmatrix}$ $\underline{t} = \begin{bmatrix} t_x \\ t_y \end{bmatrix}$

Rotation of image coordinates by angle α:

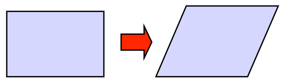
$\underline{v}' = R\,\underline{v}$ with $R = \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix}$

Scaling by factor a in x-direction and factor b in y-direction:

$\underline{v}' = S\,\underline{v}$ with $S = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$

Skewing by angle β:

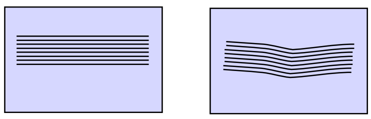
$\underline{v}' = W\,\underline{v}$ with $W = \begin{bmatrix} 1 & \tan\beta \\ 0 & 1 \end{bmatrix}$



Example of Geometry Correction by Scaling

Distortions of electron-tube cameras may be

1 - 2 % => more than 5 lines for TV images



ideal image actual image

Correction procedure may be based on

- fiducial marks engraved into optical system
- a test image with regularly spaced marks

Ideal mark positions:

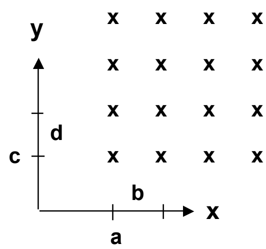
$x_{mn} = a + mb, \quad y_{mn} = c + nd$

$m = 0 \ldots M\text{-}1$

$n = 0 \ldots N\text{-}1$

Actual mark positions:

x'_{mn}, y'_{mn}



Determine a, b, c, d such that MSE (mean square error) of deviations is minimized

Minimizing the MSE

Minimize
$$E = \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} (x_{mn} - x'_{mn})^2 + (y_{mn} - y'_{mn})^2$$
$$= \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} (a + mb - x'_{mn})^2 + (c + nd - y'_{mn})^2$$

From dE/da = dE/db = dE/dc = dE/dd = 0 we get:

$$a = \frac{2}{MN(M+1)} \sum_m \sum_n (2M - 1 - 3m)\ x'_{mn}$$

$$b = \frac{6}{MN(M^2 - 1)} \sum_m \sum_n (2m - M + 1) x'_{mn}$$

$$c = \frac{2}{MN(N+1)} \sum_m \sum_n (2N - 1 - 3n)\ y'_{mn}$$

$$d = \frac{6}{MN(N^2 - 1)} \sum_m \sum_n (2n - N + 1) y'_{mn}$$

Special case M=N=2:

$a = 1/2\ (x'_{00} + x'_{01})$

$b = 1/2\ (x'_{10} - x'_{00} + x'_{11} - x'_{01})$

$c = 1/2\ (y'_{00} + y'_{01})$

$d = 1/2\ (y'_{01} - y'_{00} + y'_{11} - y'_{10})$

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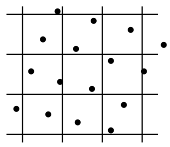
Principle of Greyvalue Interpolation

Greyvalue interpolation = computation of unknown greyvalues at locations (u´v´) from known greyvalues at locations (x´y´)



Two ways of viewing interpolation in the context of geometric transformations:

A Greyvalues at grid locations (x y) in old image are placed at corresponding locations (x´y´) in new image: g(x´y´) = g(T(x y))

=> interpolation in new image

B Grid locations (u´v´) in new image are transformed into corresponding locations (u v) in old image: g(u v) = g(T^{-1}(u´v´))

=> interpolation in old image

We will take view B:

Compute greyvalues between grid from greyvalues at grid locations.

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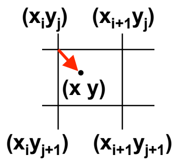
Nearest Neighbour Greyvalue Interpolation

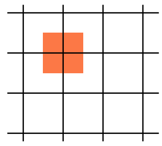
Assign to (x y) greyvalue of nearest grid location



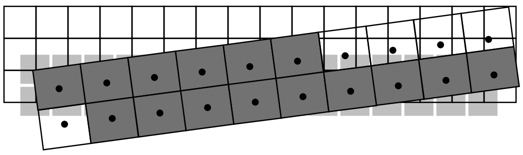
$(x_i y_j)$ $(x_{i+1} y_j)$ $(x_i y_{j+1})$ $(x_{i+1} y_{j+1})$ grid locations

(x y) location between grid with $x_i \le x \le x_{i+1}$, $y_j \le y \le y_{j+1}$

Each grid location represents the greyvalues in a rectangle centered around this location:



Straight lines or edges may appear step-like after this transformation:

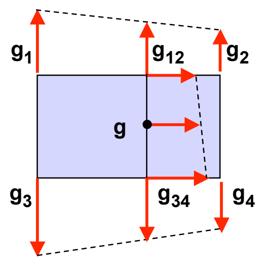


Bilinear Greyvalue Interpolation

The greyvalue at location (x y) between 4 grid points $(x_i y_j)$ $(x_{i+1} y_j)$ $(x_i y_{j+1})$ $(x_{i+1} y_{j+1})$ is computed by linear interpolation in both directions:

$$g(x,y) = \frac{1}{(x_{i+1} - x_i)(y_{j+1} - y_i)} \{ (x_{i+1} - x)(y_{j+1} - y) g(x_i y_j) + (x - x_i)(y_{j+1} - y) g(x_{i+1} y_j) + (x_{i+1} - x)(y - y_j) g(x_i y_{j+1}) + (x - x_i)(y - y_j) g(x_{i+1} y_{j+1}) \}$$

Simple idea behind long formula:
1. Compute g_{12} = linear interpolation of g_1 and g_2
2. Compute g_{34} = linear interpolation of g_3 and g_4
3. Compute g = linear interpolation of g_{12} and g_{34}

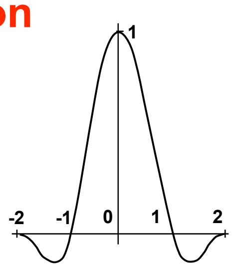


The step-like boundary effect is reduced.
But bilear interpolation may blur sharp edges.

Bicubic Interpolation

Each greyvalue at a grid point is taken to represent the center value of a local bicubic interpolation surface with cross section h_3.

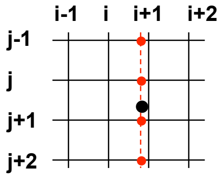
$$h_3 = \begin{cases} 1 - 2|x|^2 + |x|^3 & \text{for } 0 < |x| < 1 \\ 4 - 8|x| + 5|x|^2 - |x|^3 & \text{for } 1 < |x| < 2 \\ 0 & \text{otherwise} \end{cases}$$



cross section of interpolation kernel

The greyvalue at an arbitrary point [u, v] (black dot in figure) can be computed by

- 4 horizontal interpolations to obtain greyvalues at points [u, j-1] ... [u, j+2] (red dots), followed by
- 1 vertical interpolation (between red dots) to obtain greyvalue at [u, v].



Note:
For an image with constant geyvalues g_0 the interpolated greyvalues at all points between the grid lines are also g_0.

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