# Learning Under Uncertainty

- We want to learn models from data.

$$P(model|data) = \frac{P(data|model) \times P(model)}{P(data).}$$

- The likelihood, $P(data|model)$, is the probability that this model would have produced this data.

- The prior, $P(model)$, encodes the learning bias

# Bayesian Leaning of Probabilities

- Suppose there are two outcomes $A$ and $\neg A$. We would like to learn the probability of $A$ given some data.

- We can treat the probability of $A$ as a real-valued random variable on the interval $[0, 1]$, called *probA*.

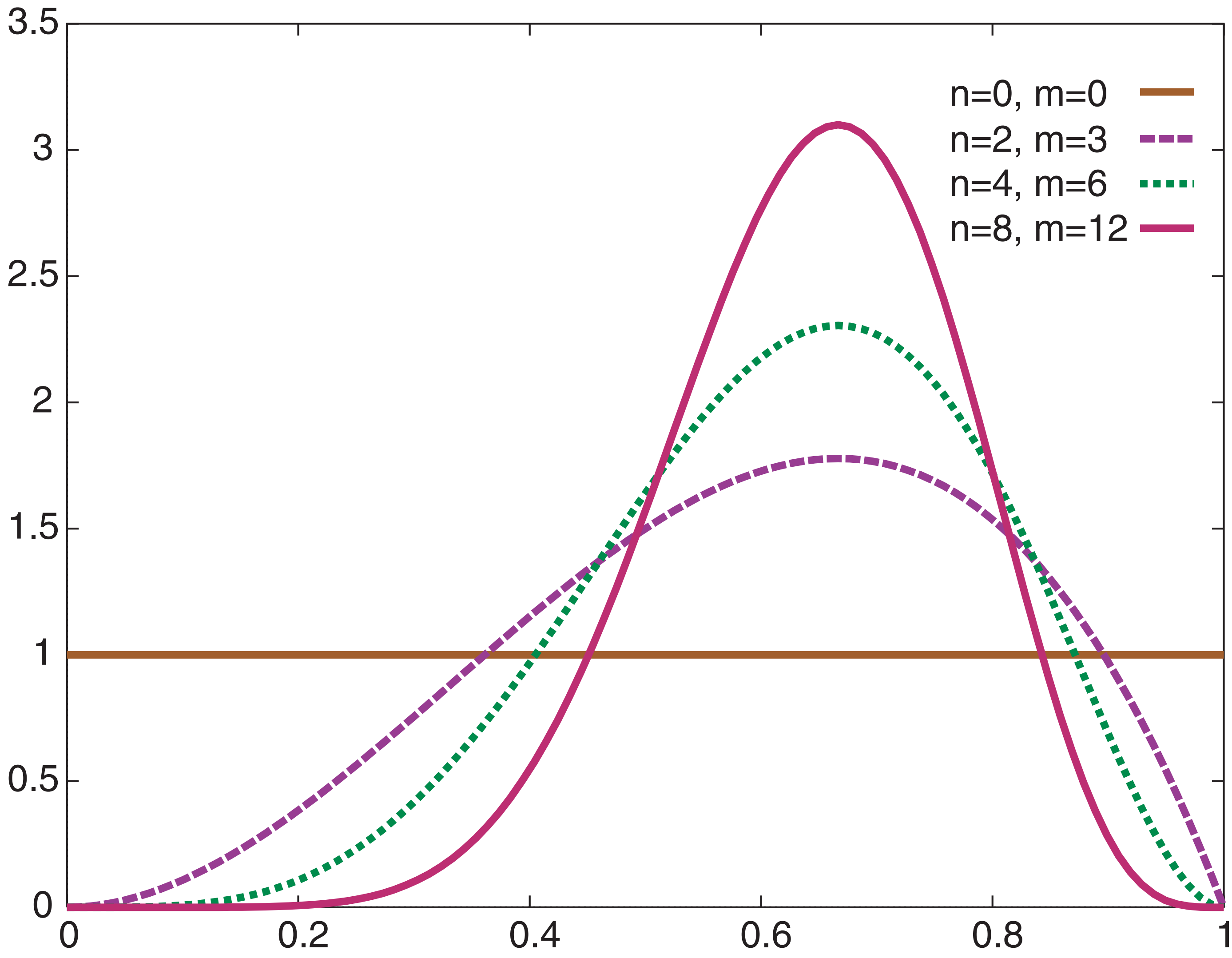
$$P(probA{=}p|data) = \frac{P(data|probA{=}p) \times P(probA{=}p)}{P(data)}$$

- Suppose the data is a sequence of $n$ $A$'s out of independent $m$ trials,

$$P(data|probA{=}p) = p^n \times (1-p)^{m-n}$$

- Uniform prior: $P(probA{=}p) = 1$ for all $p \in [0, 1]$.

# Posterior Probabilities for Different Data



# MAP model

- The maximum a posteriori probability (MAP) model is the model that maximizes $P(model|data)$. That is, it maximizes:

$$P(data|model) \times P(model)$$

- Thus it minimizes:

$$(-\log P(data|model)) + (-\log P(model))$$

which is the number of bits to send the data given the model plus the number of bits to send the model.

# Information theory overview

- A **bit** is a binary digit.
- 1 bit can distinguish 2 items
- $k$ bits can distinguish $2^k$ items
- $n$ items can be distinguished using $\log_2 n$ bits
- Can you do better?

# Information and Probability

Let's design a code to distinguish elements of $\{a, b, c, d\}$ with

$$P(a) = \frac{1}{2}, P(b) = \frac{1}{4}, P(c) = \frac{1}{8}, P(d) = \frac{1}{8}$$

Consider the code:

$$a \quad 0 \qquad\qquad b \quad 10 \qquad\qquad c \quad 110 \qquad\qquad d \quad 111$$

This code sometimes uses 1 bit and sometimes uses 3 bits.

On average, it uses

$$\begin{aligned} & P(a) \times 1 + P(b) \times 2 + P(c) \times 3 + P(d) \times 3 \\ = \quad & \frac{1}{2} + \frac{2}{4} + \frac{3}{8} + \frac{3}{8} = 1\frac{3}{4} \text{ bits.} \end{aligned}$$

The string *aacabbda* has code 00110010101110.

# Information Content

- To identify $x$, you need $-\log_2 P(x)$ bits.
- If you have a distribution over a set and want to a identify a member, you need the expected number of bits:

$$\sum_x -P(x) \times \log_2 P(x).$$

This is the information content or entropy of the distribution.

- The expected number of bits it takes to describe a distribution given evidence $e$:

$$I(e) = \sum_x -P(x|e) \times \log_2 P(x|e).$$

# Information Gain

If you have a test that can distinguish the cases where $\alpha$ is true from the cases where $\alpha$ is false, the information gain from this test is:

$$I(true) - (P(\alpha) \times I(\alpha) + P(\neg\alpha) \times I(\neg\alpha)).$$

- $I(true)$ is the expected number of bits needed before the test
- $P(\alpha) \times I(\alpha) + P(\neg\alpha) \times I(\neg\alpha)$ is the expected number of bits after the test.

# Averaging Over Models

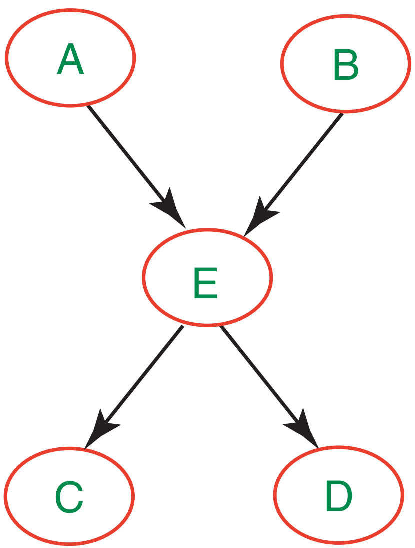
- **Idea:** Rather than choosing the most likely model, average over all models, weighted by their posterior probabilities given the data.
- If you have observed $n$ $A$'s out of $m$ trials
  - the most likely value (MAP) is $\frac{n}{m}$
  - the expected value is $\frac{n+1}{m+2}$

# Learning a Belief Network

- If you
  - know the structure
  - have observed all of the variables
  - have no missing data
- you can learn each conditional probability separately.

# Learning belief network example

Model | Data | $\rightarrow$ Probabilities




| $A$ | $B$ | $C$ | $D$ | $E$ |
|---|---|---|---|---|
| $t$ | $f$ | $t$ | $t$ | $f$ |
| $f$ | $t$ | $t$ | $t$ | $t$ |
| $t$ | $t$ | $f$ | $t$ | $f$ |
| | | $\ldots$ | | |

$P(A)$

$P(B)$

$P(E|A, B)$

$P(C|E)$

$P(D|E)$

# Learning conditional probabilities

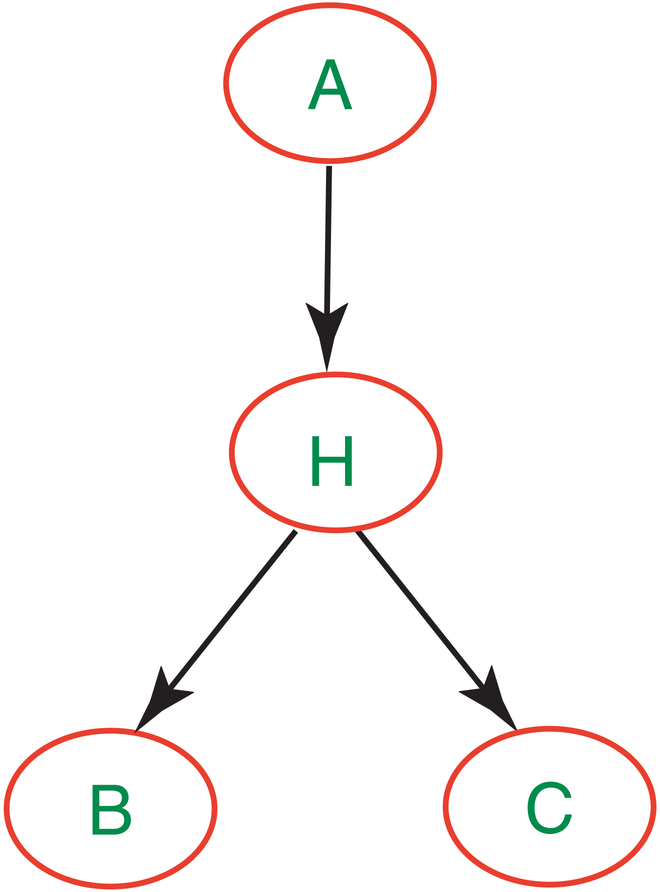
- Each conditional probability distribution can be learned separately:
- For example:

$$P(E = t | A = t \wedge B = f) = \frac{(\#\text{examples: } E = t \wedge A = t \wedge B = f) + m_1}{(\#\text{examples: } A = t \wedge B = f) + m}$$

where $m_1$ and $m$ reflect our prior knowledge.

- There is a problem when there are many parents to a node as then there is little data for each probability estimate.

# Unobserved Variables




➤ What if we had only observed values for $A$, $B$, $C$?

| $A$ | $B$ | $C$ |
|---|---|---|
| $t$ | $f$ | $t$ |
| $f$ | $t$ | $t$ |
| $t$ | $t$ | $f$ |
| | $\ldots$ | |

# EM Algorithm

**Augmented Data**

| $A$ | $B$ | $C$ | $H$ |
|---|---|---|---|
| $t$ | $f$ | $t$ | $t$ |
| $f$ | $t$ | $t$ | $f$ |
| $t$ | $t$ | $f$ | $t$ |
| | $\cdots$ | | |

E-step

M-step

**Probabilities**

$P(A)$

$P(H|A)$

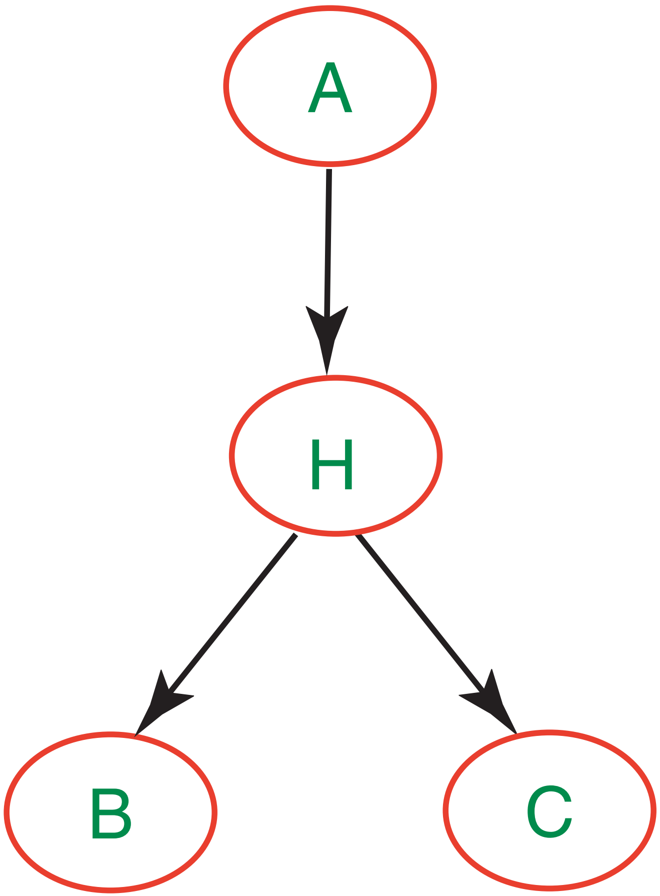
$P(B|H)$

$P(C|H)$

# EM Algorithm

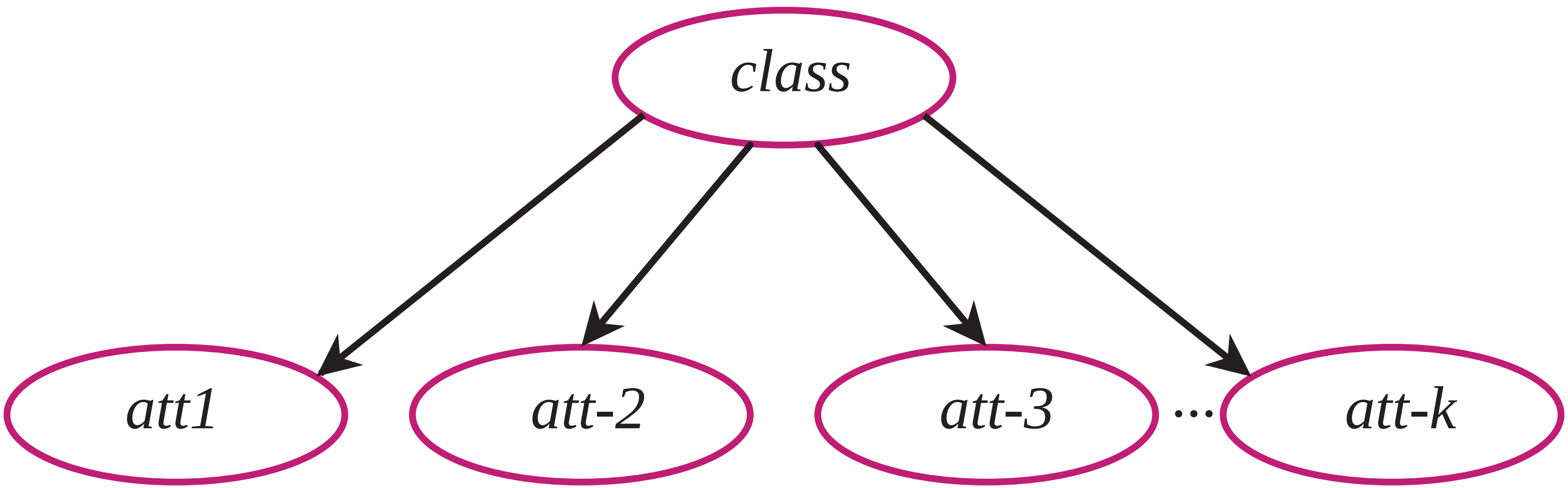
- Repeat the following two steps:
  - E-step give the expected number of data points for the unobserved variables based on the given probabilty distribution.
  - M-step infer the (maximum likelihood) probabilities from the data. This is the same as the full observable case.
- Start either with made-up data or made-up probabilities.
- EM will converge to a local maxima.

# Example Data




| A | B | C | Count |
|---|---|---|---|
| t | t | t | 143 |
| t | t | f | 329 |
| t | f | t | 57 |
| t | f | f | 271 |
| f | t | t | 87 |
| f | t | f | 66 |
| f | f | t | 23 |
| f | f | f | 24 |

# Naive Bayesian Classifier



# Unsupervised Learning

- Given a collection of data, find natural classifications.
- This can be seen as the naive Bayesian classifier with the classification unobserved.
- EM can be used to learn classification.

# Bayesian learning of decision trees

$$P(model|data) = \frac{P(data|model) \times P(model)}{P(data)}.$$

- A model here is a decision tree
- We allow for decision trees with probabilities at the leaves
- A bigger decision tree can always fit the data better
- $P(model)$ lets us encode a preference for smaller decision trees.

# Data for decision tree learning

| $att_1$ | $att_2$ | *class* | count |
|---|---|---|---|
| t | t | c1 | 10 |
| t | t | c2 | 3 |
| t | f | c1 | 5 |
| t | f | c2 | 12 |
| f | t | c1 | 7 |
| f | t | c2 | 14 |
| f | f | c1 | 8 |
| f | f | c2 | 1 |

# Probabilities From Experts

- Bayes rule lets us combine expert knowledge with data

$$P(model|data) = \frac{P(data|model) \times P(model)}{P(data)}.$$

- The experts prior knowledge of the model (i.e., $P(model)$) can be expressed as a pair $\langle n, m \rangle$ that can be interpreted as though they had observed $n$ $A$'s out of $m$ trials.

- This estimate can be combined with data.

- Estimates from multiple experts can be combined together.