Universität Hamburg
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MIN-Fakultät
Fachbereich Informatik
Arbeitsbereich SAV/BV (KOGS)

Image Processing 1 (IP1)
Bildverarbeitung 1

Lecture 8 – Image Compression 1

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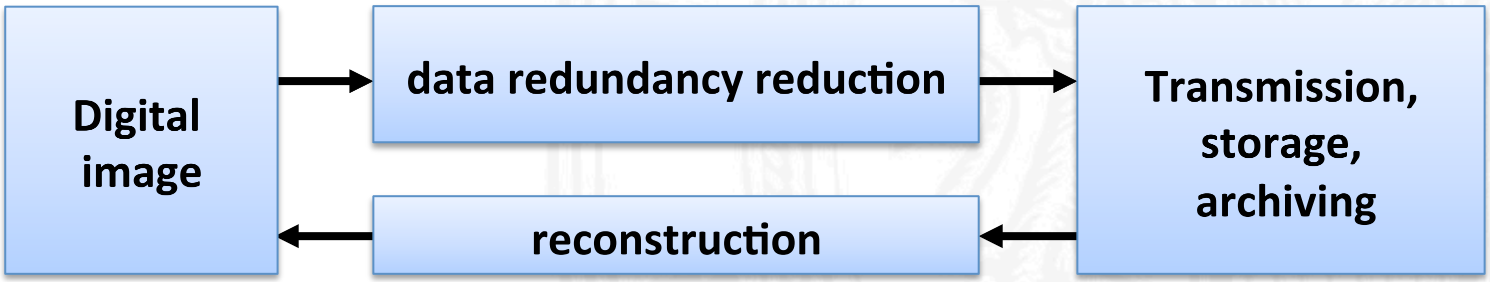
Image Data Compression

Data compression allows to save storage space as well as transmissian capacity for digital/discrete data. Therefore, the data will be rearranged in a new structure with needs less storage than the former one.

Image data compression is important for:

• Image archives	e.g large satellite images
• Image transmission	e.g. over the internet
• Multimedia applications	e.g. desktop editing

Image data compression exploits redundancy for more efficient coding:



Different Kinds of Compression

Lossless compression:

- Original data may be reconstructed **exactly** from the compressed data.
- Examples:
  - File compression, e.g. ZIP, GZ etc.
  - Image formats: TIFF and PNG

Lossy compression:

- Original daten may be reconstructed **approximately** from the compressed data.
- Examples:
  - JPEG image files
  - DVDs and Blu-Rays
  - MP3 audio files

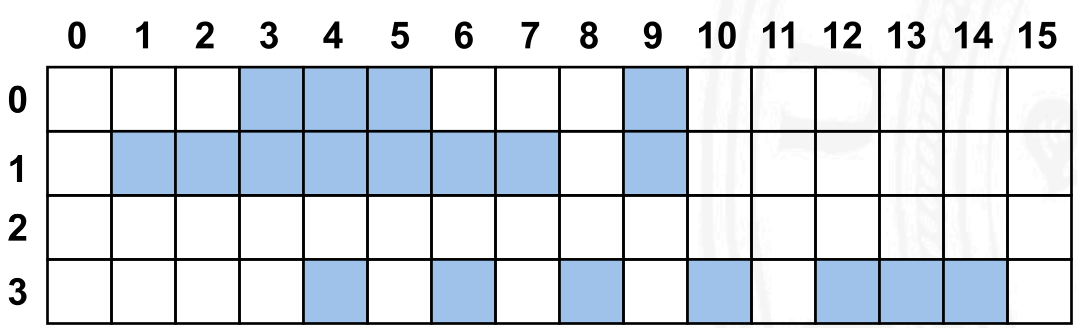
Run Length Coding

Images with repeating greyvalues along rows (or columns) can be compressed by storing "runs" of identical greyvalues in the format:

greyvalue1	repetition1	greyvalue2	repetition2	...

For B/W images (e.g. fax data) another run length code is used:

row #	column# run1 begin	column# run1 end	column# run2 begin	column# run2 end	...



Run length code:

(0 3 5 9 9)
(1 1 7 9 9)
(3 4 4 6 6 8 8 10 10 12 14)

Probabilistic Data Compression

A discrete image encodes information redundantly if

1. the greyvalues of individual pixels are not equally probable
2. the greyvalues of neighbouring pixels are correlated

Information Theory provides limits for minimal encoding of probabilistic information sources.

Redundancy of the encoding of individual pixels with G greylevels each:

$$r = b - H$$

$b = \lceil \log_2 G \rceil$ = **number of bits used for each pixel**

$$H = \sum_{g=0}^{G-1} P(g) \log_2 \frac{1}{P(g)}$$

H = **<u>entropy</u> of pixel source**

= **mean number of bits required to encode information of this source**

The entropy of a pixel source with equally probable greyvalues is equal to the number of bits required for coding.

Huffman Coding I

The Huffman coding scheme provides a variable-length code with minimal average code-word length, i.e. least possible redundancy, for a discrete message source.
(Here messages are greyvalues)

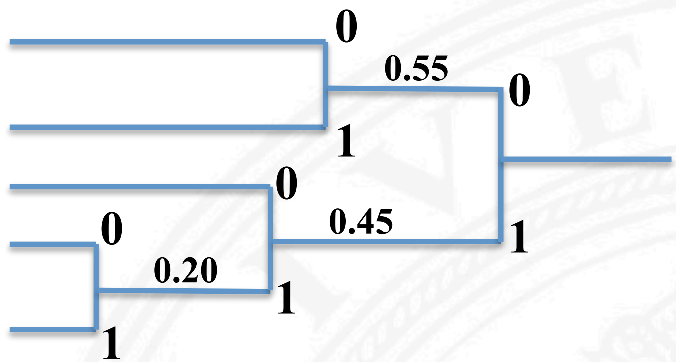
Algorithm:

1. Sort messages along increasing probabilities such that $g^{(1)}$ and $g^{(2)}$ are the least probable messages
2. Assign 1 to code word of $g^{(1)}$ and 0 to the code word of $g^{(2)}$.
3. Merge $g^{(1)}$ und $g^{(2)}$ to one new message by adding their probabilities.
4. Repeat steps 1 – 4 until a single message is left.

Huffman Coding II

Example:

Message	Probability
$g^{(5)}$	0.30
$g^{(4)}$	0.25
$g^{(3)}$	0.25
$g^{(2)}$	0.10
$g^{(1)}$	0.10



Resultierende Codierungen:

Message	Probability	Coding
$g^{(5)}$	0.30	00
$g^{(4)}$	0.25	01
$g^{(3)}$	0.25	10
$g^{(2)}$	0.10	110
$g^{(1)}$	0.10	111

Entropy: $H = 2.185$

↓

Mean code word length: 2.2

Statistical Dependence

An image may be modelled as a set of statistically dependent random variables with a multivariate distribution $p\left(\vec{x}\right) = p\left(x_1, x_2, \ldots, x_N\right)$

Often the exact distribution is unknown and only correlations can be (approximately) determined.

<u>Correlation</u> of two variables:

$$E\left[x_i x_j\right] = c_{ij}$$

$$E\left[\vec{x}\,\vec{x}^T\right] = \begin{bmatrix} c_{11} & c_{12} & c_{13} & \cdots \\ c_{21} & c_{22} & c_{23} & \cdots \\ c_{31} & c_{32} & c_{33} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

Correlation matrix

<u>Covariance</u> of two variables:

$$E\left[(x_i - \mu_i)(x_j - \mu_j)\right] = v_{ij} \quad \text{with } \mu_k = \text{mean of } x_k$$

$$E\left[(\vec{x} - \vec{\mu})(\vec{x} - \vec{\mu})^T\right] = \begin{bmatrix} v_{11} & v_{12} & v_{13} & \cdots \\ v_{21} & v_{22} & v_{23} & \cdots \\ v_{31} & v_{32} & v_{33} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

Covariance matrix

Attention: Uncorrelated variables need not be statistically independent:

$$E\left[x_i x_j\right] = 0 \quad \not\Rightarrow \quad p\left(x_i x_j\right) = p\left(x_i\right) \cdot p\left(x_j\right)$$

But: For Gaussian random variables, uncorrelatedness implies statistical independence.

Karhunen-Loève Transform

(also known as Hotelling Transform or Principal Components Analysis)

Determine uncorrelated variables $\vec{y}$ from correlated variables $\vec{x}$ by a linear transformation.

$$y = A(\vec{x} - \vec{\mu})$$

$$E\left[\vec{y}\,\vec{y}^T\right] = A\,E\left[(\vec{x} - \vec{\mu})(\vec{x} - \vec{\mu})^T\right]A^T = AVA^T = D$$

***D* is a diagonal matrix**

- An orthonormal matrix A which diagonalizes the real symmetric covariance matrix V always exists.
- A is the matrix of eigenvectors of V, D is the matrix of corresponding eigenvalues.

Reconstruction of $\vec{x}$ from $\vec{y}$ using: $\vec{x} = A^T\vec{y} + \vec{\mu}$

Note: If $\vec{x}$ is viewed as a point in n-dimensional Euclidean space, then A defines a rotated coordinate.

Kompression and Rekonstruktion mit der Karhunen-Loève Transformation

Wir nehmen an, dass die Eigenwerte λ_i und die zugehörigen Eigenvektoren von A in absteigender Reihenfolge sortiert sind: $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_N$

$$D = \begin{bmatrix} \lambda_1 & 0 & 0 & \cdots \\ 0 & \lambda_2 & 0 & \cdots \\ 0 & 0 & \lambda_3 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

Eigenvektoren $\vec{a}$ und Eigenwerte λ sind definiert durch $V\vec{a} = \lambda\vec{a}$ und können bestimmt werden, indem man $\det(V - \lambda I) = 0$ löst.

Zum Bestimmen der Eigenwerte von reellen symmetrischen Matrizen gibt es spezielle Verfahren.

$\vec{x}$ kann in einen K-dimensionalen Vektor $\vec{y}_K$, $K < N$ transformiert werden, mit einer Transformationsmatrix A_K, die nur die ersten K Eigenvektoren von A enthält, korrespondierend zu den K größten Eigenwerten:

$$\vec{y}_K = A_K(\vec{x} - \vec{\mu})$$

Die angenäherte Rekonstruktion $\vec{x}'$ minimiert den mittleren quadratischen Fehler (MSE – mean square error) einer Repräsentation mit K Dimensionen:

$$\vec{x}' = A_K^T y_K + \vec{\mu}$$

Deshalb kann $\vec{y}_K$ zur (verlustbehafteten) Datenkomprimierung verwendet werden.

Compression and Reconstruction using the Karhunen-Loève Transform

Let us assume that the Eigenvalues λ_i and the corresponding Eigenvectors of A are sorted descending: $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_N$

$$D = \begin{bmatrix} \lambda_1 & 0 & 0 & \cdots \\ 0 & \lambda_2 & 0 & \cdots \\ 0 & 0 & \lambda_3 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

Eigenvectors $\vec{a}$ and Eigenvalues λ are defined by $V\,\vec{a} = \lambda\,\vec{a}$ and my be estimated by solving:
$\det\left(V - \lambda I\right) = 0$ **.**
There exist fast solution methods for the Eigevalues of real symmetric matrices.

$\vec{x}$ may be transformatin into a K-dimensional vector $\vec{y}_K$, with $K < N$ using a Transformationsmatrix A_K, which contains only the first K Eigenvectors of A, which correspond to the K largest Eigenvalues:

$$\vec{y}_K = A_K\left(\vec{x} - \vec{\mu}\right)$$

The approximate reconstruction $\vec{x}'$ minimizes the – mean square error (MSE) of a representation wit K dimensions:

$$\vec{x}' = A_K^T\, y_K + \vec{\mu}$$

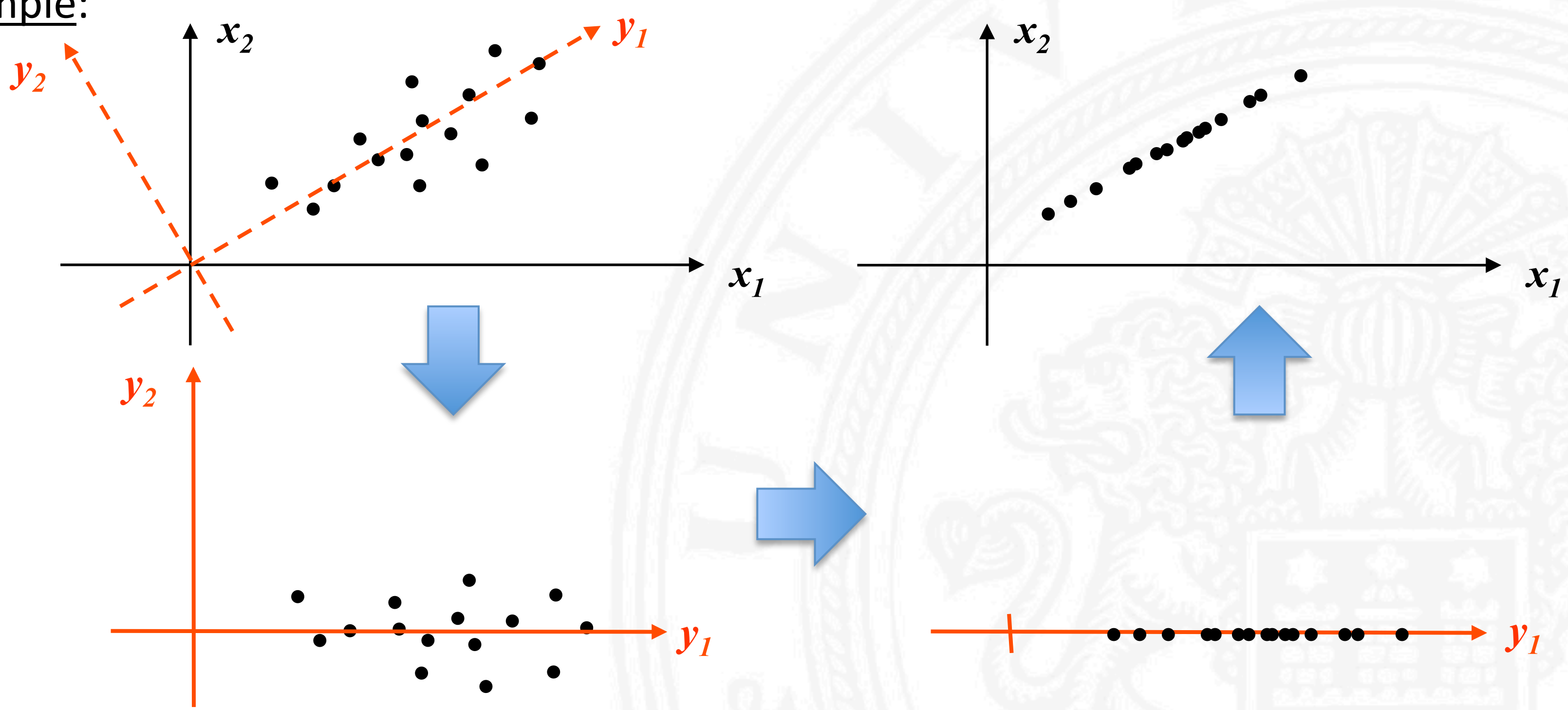
Thus, $\vec{y}_K$ is representing a lossy data compression .

Illustration of Minimum-loss Dimension Reduction

Using the Karhunen-Loève transform, data compression is achieved by

- changing (rotating) the coordinate system
- omitting the least informative dimension(s) in the new coodinate system

Example:



Numerical Example for Karhunen-Loève Compression

Given: $N = 3$

$$\vec{x}^T = (x_1 \;\; x_2 \;\; x_3) \qquad V = \begin{pmatrix} 2 & -0.866 & -0.5 \\ -0.866 & 2 & 0 \\ -0.5 & 0 & 2 \end{pmatrix}$$

$$\vec{m} = 0$$

Eigenvalues and eigenvectors

$$\det(V - \lambda I) = 0 \quad \Rightarrow \quad \lambda_1 = 3,\;\; \lambda_2 = 2,\;\; \lambda_3 = 1 \qquad V = \begin{pmatrix} 0.707 & 0 & 0.707 \\ -0.612 & 0.5 & 0.621 \\ -0.354 & -0.866 & 0.354 \end{pmatrix}$$

Compression into K=2 dimensions

$$\vec{y}_2 = A_2\,\vec{x} = \begin{pmatrix} 0.707 & -0.612 & -0.354 \\ 0 & 0.5 & -0.866 \end{pmatrix} \vec{x}$$

Reconstruction from compressed values

$$\vec{x}' = A_2^T\,\vec{y} = \begin{pmatrix} 0.707 & 0 \\ -0.612 & 0.5 \\ -0.354 & -0.866 \end{pmatrix} \vec{y}$$

Note the discrepancies between the original and the approximated values:

$x_1' = 0,5\, x_1 - 0,43\, x_2 - 0,25\, x_3$

$x_2' = -0,085\, x_1 - 0,625\, x_2 + 0,39\, x_3$

$x_3' = 0,273\, x_1 + 0,39\, x_2 + 0,25\, x_3$