Universität Hamburg
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**MIN-Fakultät**
**Fachbereich Informatik**
Arbeitsbereich SAV/BV (KOGS)

# Image Processing 1 (IP1)
# Bildverarbeitung 1

## Lecture 15 – Pattern Recognition

Winter Semester 2014/15

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# What is "Pattern Recognition"?

The term "Pattern Recognition" ("Mustererkennung") is used for

**Methods for classifying unknown objects based on feature vectors (narrow sense meaning of Pattern Recognition)**

**Methods or analyzing signals and recognizing interesting patterns (wide sense meaning of Pattern Recognition)**

Pattern recognition can be applied to all kinds of signals, e.g.

- images
- acoustic signals
- seismographic signals
- tomographic data etc.

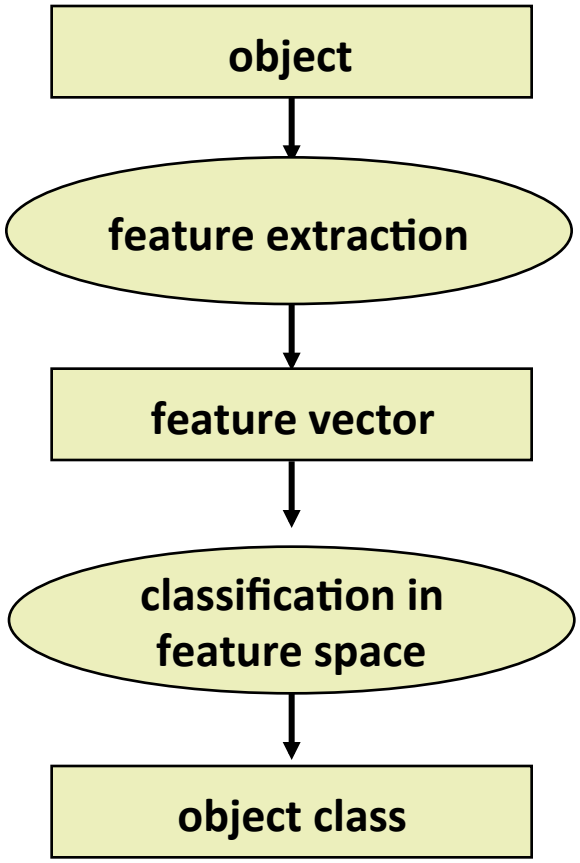
The following section deals with Pattern Recognition in the narrow sense.

(see Duda and Hart, Pattern Classification and Scene Analysis, Wiley 73)

# Introductory Example: Where is Wally?

(c) Martin Handford, Walker Books

# Basic Terminology for Pattern Recognition



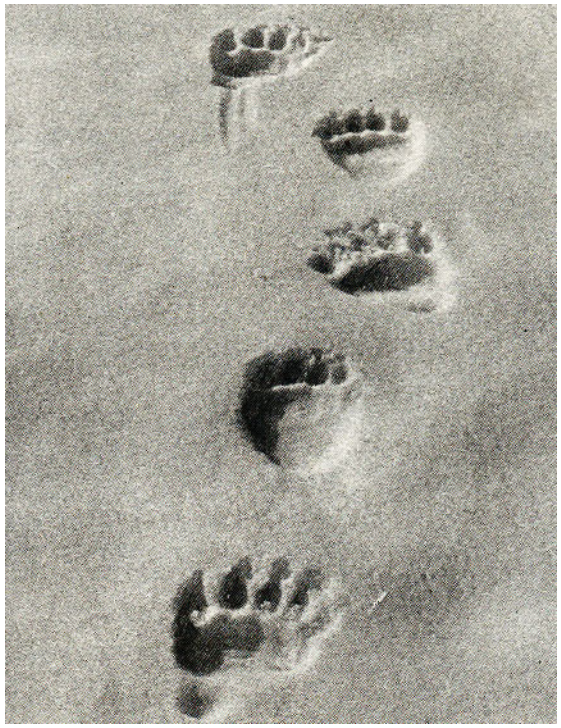

| | |
|---|---|
| $K$ | classes $\omega_1 \ldots \omega_K$ |
| $N$ | dimension of feature space |
| $\vec{x}^T = (x_1 \; x_2 \; \ldots \; x_N)$ | feature vector |
| $\vec{y}^T = (y_1 \; y_2 \; \ldots \; y_N)$ | prototype (feature vector with known class membership) |
| $\vec{y}_i^{(k)}$ | i-th prototyp of class k |
| $M_k$ | number of prototypes for class k |
| $g_k(\vec{x})$ | discriminant function for class k |

**Problem: Determine $g_k(\vec{x})$ such that**

$$\forall_{\vec{x} \in \omega_k} \; \forall_{k \neq j} \; g_k(\vec{x}) > g_j(\vec{x})$$

# Example: Animal Footprints

**Bear**



**Hare**

**Wolf**

**What features can be used to distinguish the 3 footprint classes?**

# A Feature Space for Footprints


$\omega_1$ = wolf

$\omega_2$ = bear

$\omega_3$ = hare

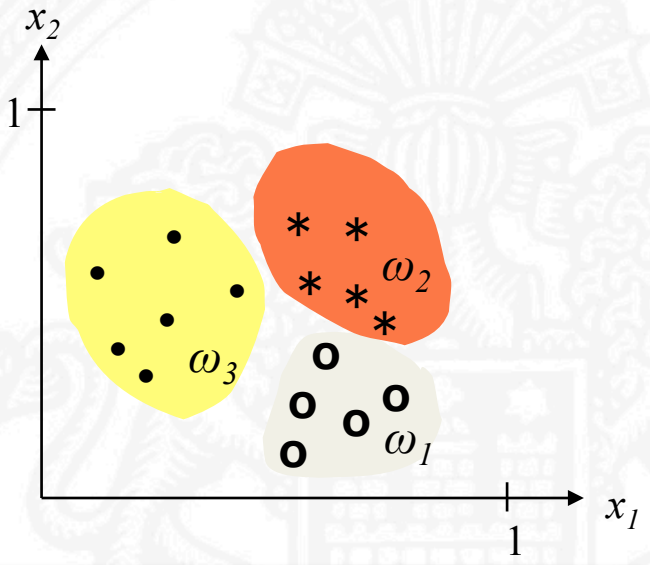
$c$ = „circumference“ = $2(w+h)$

$a$ = „area“ = $w \times h$

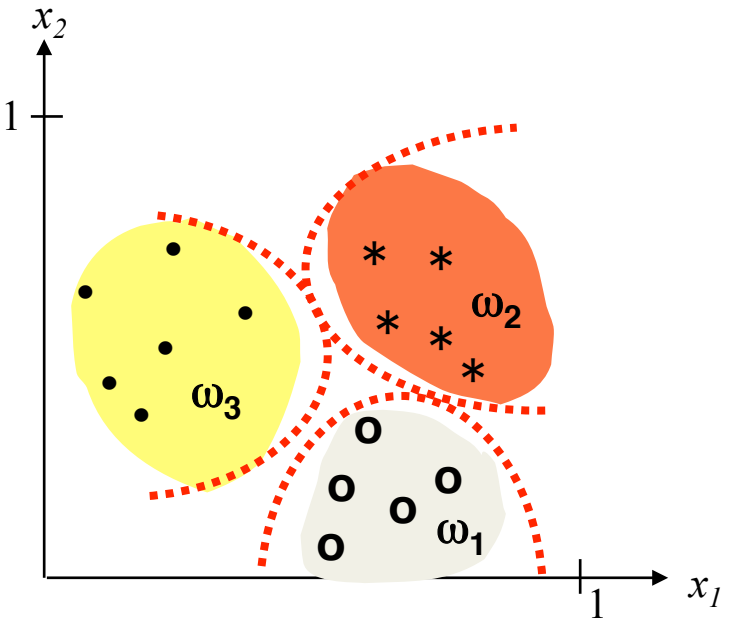
$pa$ = „print area“ = $card(\{ g(x,y) > 0\})$

$x_1$ = "squareness" = $\frac{a}{c^2}$

$x_2$ = "solidness" = $\frac{pa}{c}$



# Discriminant Functions for Footprints

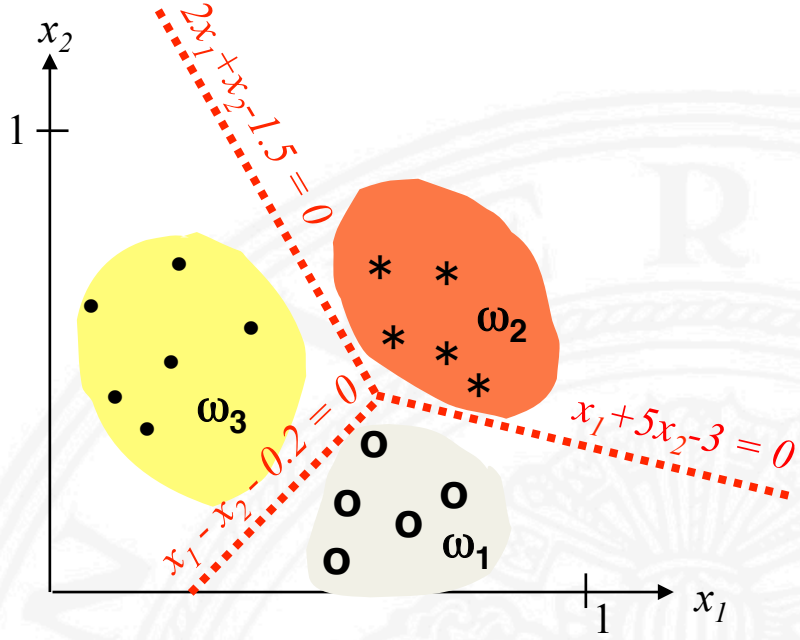



**Quadratic discriminant functions:**

$g_1 = -9x_1^2 + 10.8x_1 - x_2 - 2.84$

$g_2 = x_1 + 20x_2^2 - 28x_2 + 9.4$

$g_3 = -x_1 + 5.6x_2^2 - 5.6x_2 - 1$




**Piecewise linear discriminant functions:**

$g_1 = 1$ if $(x_1-x_2-0.2 > 0) \wedge (x_1+5x_2-3 < 0)$ else 0

$g_2 = 1$ if $(x_1+5x_2-3 > 0) \wedge (2x_1+x_2-1.5 > 0)$ else 0

$g_3 = 1$ if $(2x_1+x_2-1.5 < 0) \wedge (x_1-x_2-0.2 < 0)$ else 0

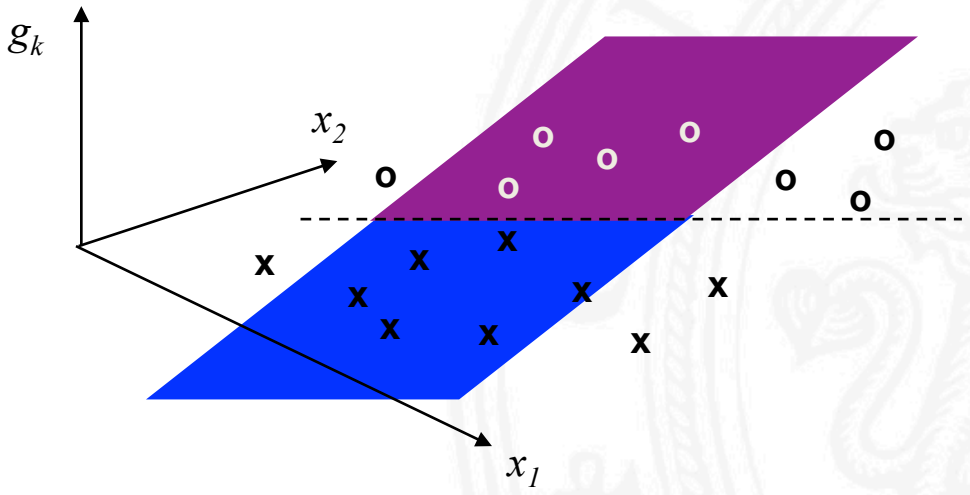
# Linear Discriminant Functions

**Linear discriminant functions are attractive because they can be**

- **easily determined from prototypes**
- **easily analyzed**
- **easily evaluated**

Basic form of linear discriminant function:

$$g_k\left(\vec{x}\right) = \left(\vec{w}_k\right)^T \vec{x} + w_{k_0}$$



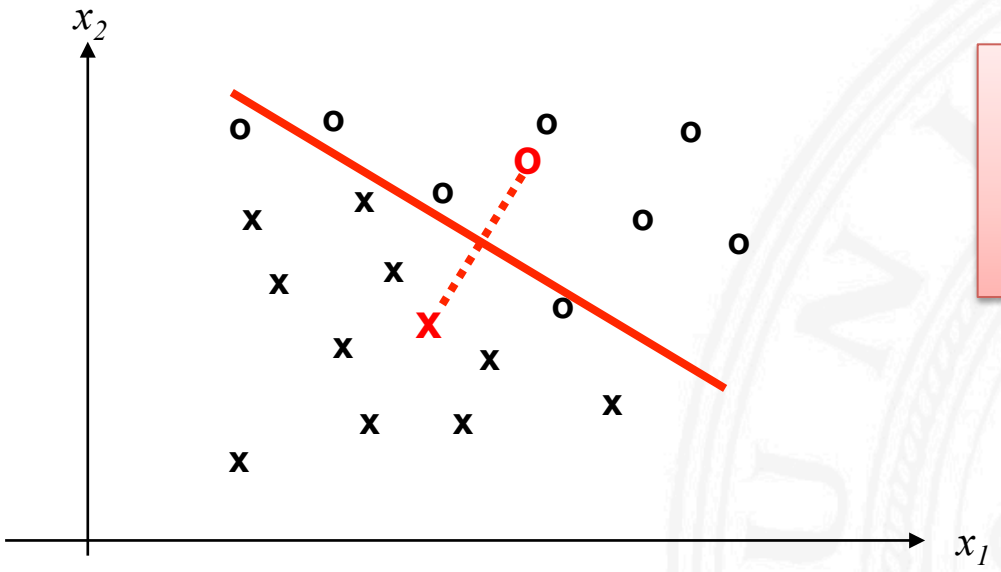

For $N$=2 the discriminant function is a 3D plane

boundary line:

$$g_k\left(\vec{x}\right) = \left(\vec{w}_k\right)^T \vec{x} + w_{k_0} = 0$$

# Class Average
# Minimal Distance Classification

- Represent prototypes by class averages
- Assign object to class with minimum distance between object and class average

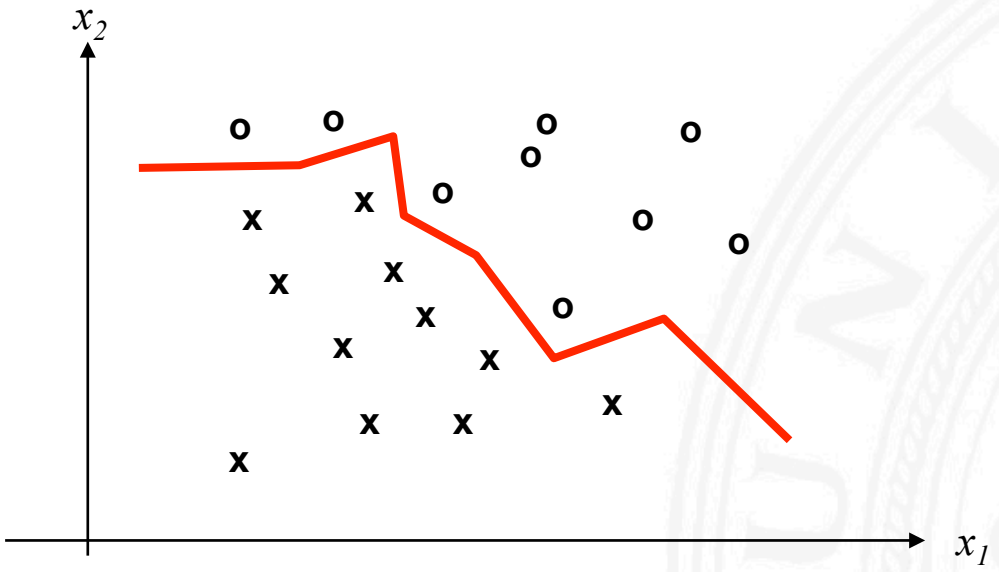



**For a 2-class problem, the minimal distance criterion always results in a linear discriminant function!**

**Class average minimal distance classification may not separate prototypes even if they are linearly separable!**

# Nearest Neighbour Classification

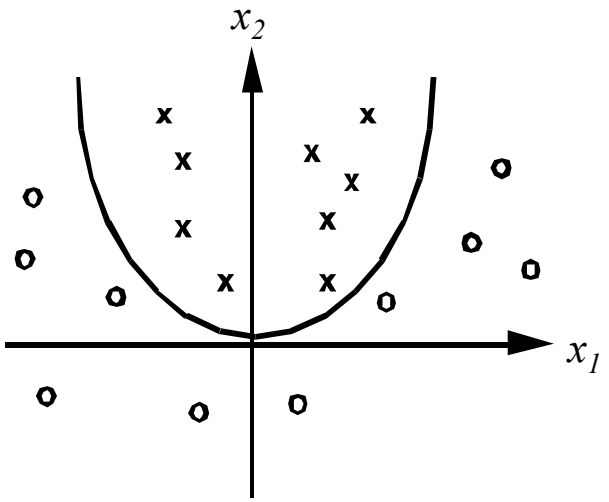
**Assign object to class with nearest prototype**




**Piece-wise linear discriminant function**

**The nearest neighbour criterion classifies all prototypes correctly (except equal prototypes of different classes). The decision regions are not necessarily coherent.**

# Generalized Linear Discriminant Functions




**Example:**

Prototypes are not linearly separable

A quadratic discriminant function may work:

$$g_k(\vec{x}) = a_1 x_1 + a_1 x_1 + b_{11}(x_1)^2 + b_{22}(x_2)^2 + b_{12} x_1 x_2 + c$$

with $\vec{x}^T = (x_1 \;\; x_2)$

**Transformation of prototypes into higher-dimensional feature space may allow linear discriminant functions.**

Transformation for the example: $z_1 = x_1 \;,\; z_2 = x_2 \;,\; z_3 = (x_1)^2 \;,\; z_4 = (x_2)^2 \;,\; z_5 = x_1 x_2$

Linear discriminant function in z-space: $g_k(\vec{z}) = a_1 z_1 + a_2 z_2 + a_3 z_3 + a_4 z_4 + a_5 z_5 + c$

Advantage: Linear separation algorithms may be applied

Disadvantage: Dimensionality of feature space is drastically increased

# Linear Discriminant Functions for 2-Class Problems

Normalize prototypes such that

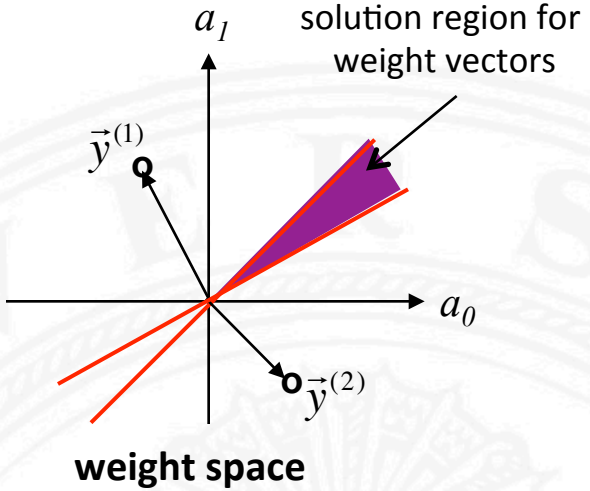
$$\vec{y}^T = (1 \; y_1 \; y_2 \; \ldots \; y_N)$$

Discriminant function $g$ can be expressed as

$$g(\vec{x}) = \vec{a}^T \vec{x} \quad \text{with} \quad \vec{a}^T = (a_0 \; a_1 \; \ldots \; a_N)$$

Prototypes of class $\omega_2$ are negated such that

$\vec{a}^T \vec{y} > 0$ → correct classification of both classes




**weight space**

Solution region in weight space (if it exists) is the space at the positive side of all hyperplanes $\vec{a}^T \vec{y} = 0$. Any weight vector $\vec{a}$ in this solution region gives a correct discriminant function.

Possible further constraints on solution vector a:

$$\|\vec{a}\| = 1 \;\wedge\; \forall_{\vec{y}} \vec{a}^T \vec{y} > b$$

$b$ is "margin", i.e. minimal distance of a correctly classified point from the hyperplanes defined by the prototypes.

# Perceptron Learning Rule

A solution vector a can be determined iteratively by minimizing a criterion function $J(\vec{a})$ by gradient descent.

**Perceptron criterion function:**

$$J_p(\vec{a}) = \sum_{y \in B} \left(-\vec{a}^T \vec{y}\right)$$

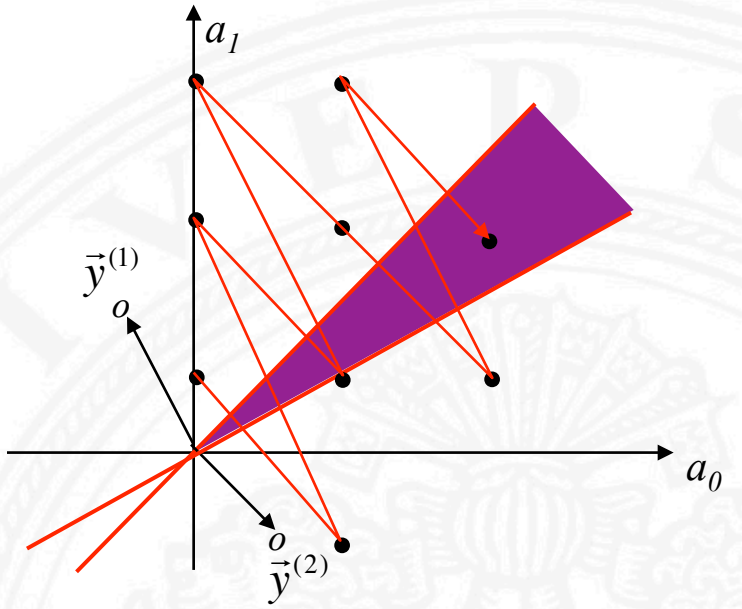
with $B$ = {all misclassified prototypes}

**Basic gradient descent algorithm:**

Gradient: $\nabla J_p(\vec{a}) = \sum_{y \in B} \left(-\vec{y}\right)$

Step: $\vec{a}_{k+1} = \vec{a}_k + \rho_k \sum_{y \in B} \left(\vec{y}\right)$

Weight vector $\vec{a}$ is modified in negative gradient direction!




iterations viewed in weight space

**Example (see illustration) with:**

$\vec{y}_1 = \begin{pmatrix}-1 & 2\end{pmatrix}^T$ , $\vec{y}_2 = \begin{pmatrix}-1 & 1\end{pmatrix}^T$ , $\rho = 2$

| $k$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|---|---|---|---|---|---|---|---|---|---|
| $\vec{a}_k$ | $\begin{matrix}0\\1\end{matrix}$ | $\begin{matrix}2\\-1\end{matrix}$ | $\begin{matrix}0\\3\end{matrix}$ | $\begin{matrix}2\\1\end{matrix}$ | $\begin{matrix}0\\5\end{matrix}$ | $\begin{matrix}2\\3\end{matrix}$ | $\begin{matrix}4\\1\end{matrix}$ | $\begin{matrix}2\\5\end{matrix}$ | $\begin{matrix}4\\3\end{matrix}$ |

solution

# Minimizing the Discriminant Criterion

General form of gradient descent:

$$\vec{a}_{k+1} = \vec{a}_k - \rho_k \nabla J(\vec{a}_k) \quad \text{with} \quad \nabla J(\vec{a}_k)^T = \left( \frac{\partial J}{\partial a_0} \; \frac{\partial J}{\partial a_1} \; \ldots \; \frac{\partial J}{\partial a_N} \right)$$

One can determine the optimal $\rho_k$ which achieves the minimal $J(\vec{a}_{k+1})$ at the $k$th step by approximating $J(\vec{a})$ with a second-order Taylor series expansion:

$$J(\vec{a}_k) \approx J(\vec{a}_{k+1}) + \nabla^T J(\vec{a}_k)(\vec{a} - \vec{a}_k) + \tfrac{1}{2}(\vec{a} - \vec{a}_k)^T D(\vec{a}_k)(\vec{a} - \vec{a}_k)$$

where $D(\vec{a}_k)$ is the matrix of second derivatives $\frac{\partial^2 J}{a_i \partial a_j}$ evaluated at $\vec{a}_k$.

Using the iteration rule:

$$J(\vec{a}_{k+1}) \approx J(\vec{a}_k) - \rho_k \left\| \nabla^T J(\vec{a}_k) \right\|^2 + \tfrac{1}{2} \left( \rho_k \right)^2 \nabla J(\vec{a}_k)^T D(\vec{a}_k) \nabla J(\vec{a}_k)$$

The minimizing $\rho_k$ is:

$$\rho_k = \frac{\left\| \nabla^T J(\vec{a}_k) \right\|^2}{\nabla J(\vec{a}_k)^T D(\vec{a}_k) \nabla J(\vec{a}_k)}$$

**Newton´s algorithm** is an alternative:
Choose $\vec{a}_{k+1}$ which minimizes $J(\vec{a})$ in the Taylor series approximation.

$$\vec{a}_{k+1} = a_k - D^{-1} \nabla J\left(a_k\right)$$

# Quadratic Criterion Function

**Quadratic criterion function:**

$$J_q(\vec{a}) = \sum_{y \in B} \left(\vec{a}^T \vec{y}\right)^2 \qquad \text{with } B = \{\text{all samples where } \vec{a}^T \vec{y} \le 0\}$$

Problems:

- slow convergence close to boundaries $\vec{a}^T \vec{y} \approx 0$
- dominated by long sample vectors $\vec{y}$

**Normalized quadratic criterion function:**

$$J_r(\vec{a}) = \frac{1}{2} \sum_{y \in B} \frac{\left(\vec{a}^T \vec{y} - b\right)^2}{\left\|\vec{y}\right\|^2} \qquad \text{with } B = \{\text{all samples where } \vec{a}^T \vec{y} < b\}$$

Gradient: $\nabla J_r(\vec{a}) = \sum_{y \in B} \frac{\vec{a}^T \vec{y} - b}{\left\|\vec{y}\right\|^2} \vec{y}$

Iteration rule: $\vec{a}_{k+1} = \vec{a}_k + \rho_k \sum_{y \in B} \frac{b - \vec{a}^T \vec{y}}{\left\|\vec{y}\right\|^2} \vec{y}$

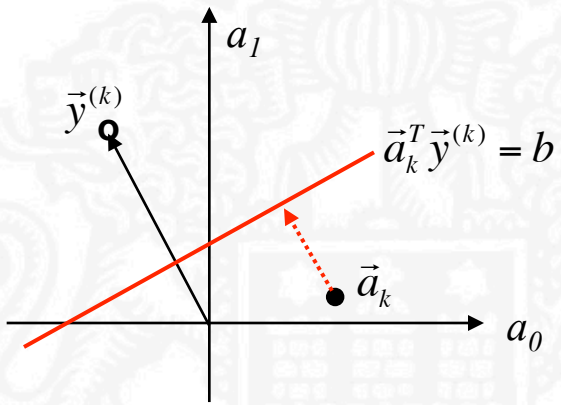
# Relaxation Rule

If corrections based on the normalized quadratic criterion are performed for each single sample, one gets the "relaxation rule":

$$\vec{a}_{k+1} = \vec{a}_k + \rho \frac{b - \vec{a}_k^T \vec{y}^{(k)}}{\left\| \vec{y}^{(k)} \right\|^2} \vec{y}^{(k)} \qquad \text{where} \quad \forall_k \; \vec{a}_k^T \vec{y}^{(k)} < b$$

Distance from $\vec{a}_k$ to hyperplane $\vec{a}_k^T \vec{y}^{(k)} = b$ is: $\dfrac{b - \vec{a}_k^T \vec{y}^{(k)}}{\left\| \vec{y}^{(k)} \right\|^2}$

For $\rho = 1$, the iteration rule calls for moving $\vec{a}_k$ directly to the hyperplane
→"relaxation" of tension in inequality $\vec{a}_k^T \vec{y}^{(k)} < b$




**Typical values:** $0 < \rho < 2$
$\rho < 1$ "underrelaxation"
$\rho > 1$ "overrelaxation"

# Minimum Squared Error

New criterion function for all samples:

Find $\vec{a}$ such that $\vec{a}^T \vec{y}_i = b_i$ with $b_i$ = some positive constant

In matrix notation: $Y\vec{a} = \vec{b}$ with $Y = \begin{pmatrix} \vec{y}_1{}^T \\ \vec{y}_2{}^T \\ \vdots \\ \vec{y}_M{}^T \end{pmatrix}$ and $\vec{y}_i = \begin{pmatrix} y_{i_1} \\ y_{i_2} \\ \vdots \\ y_{i_{nN}} \end{pmatrix}$

In general, $M >> N$ and $Y^{-1}$ does not exist, hence $\vec{a} = Y^{-1}\vec{b}$ is no solution.

Classical solution technique: Minimize squared error criterion:

$$J_s(\vec{a}) = \left\| Y\vec{a} - \vec{b} \right\|^2 = \sum \left( \vec{a}^T \vec{y}_i - b_i \right)^2$$

Closed-form solution by setting the gradient equal to 0.

$$\nabla J_s(\vec{a}) = 2Y^T \left( Y\vec{a} - \vec{b} \right) = 0 \Rightarrow \vec{a} = \left( Y^T Y \right)^{-1} Y^T \vec{b}$$

if $(Y^TY)^{-1}Y^T$ is nonsingular

**pseudoinverse of Y**

# Ho-Kashyap Procedure

The MSE solution $\vec{a} = \left(Y^T Y\right)^{-1} Y^T \vec{b}$ does not necessarily provide a separating hyperplane if the classes are linearly separable, because $\vec{b}$ is chosen arbitrarily.

**Ho-Kashyap** algorithm searches for $\vec{a}$ and $\vec{b}$ such that $Y\vec{a} = \vec{b} > \vec{0}$ by minimizing $J_s$ w.r.t. $\vec{a}$ and $\vec{b}$:

1. Iterate over $\vec{a}$ by choosing $\vec{a}_k = \left(Y^T Y\right)^{-1} Y^T \vec{b}_k$
2. Iterate over $\vec{b}$ by choosing $\vec{b}_1 > \vec{0}$:

$$\vec{b}_{k+1} = b_k + 2\rho\vec{e}_k^{\,+} \qquad 0 < \rho < 1$$

with $\vec{e}_k = Y\vec{a}_k - \vec{b}_k$ error vector

$\vec{e}_k^{\,+} = \frac{1}{2}\left(\vec{e}_k + |\vec{e}_k|\right)$ positive part of $\vec{e}_k$

Ho-Kashyap iteration over $\vec{b}$ generates sequence of margin vectors $\vec{b}$ which

- minimizes squared error criterion
- gives only positive margins $\vec{b} > \vec{0}$

**For linearly separable classes and $0 < \rho < 1$, the Ho-Kashyap algorithm will converge in a finite number of steps.**

# Discrimination with Potential Functions

**Idea**: Electrostatic potential centered at each prototype may sum up to a useful discriminant function

**Example**:

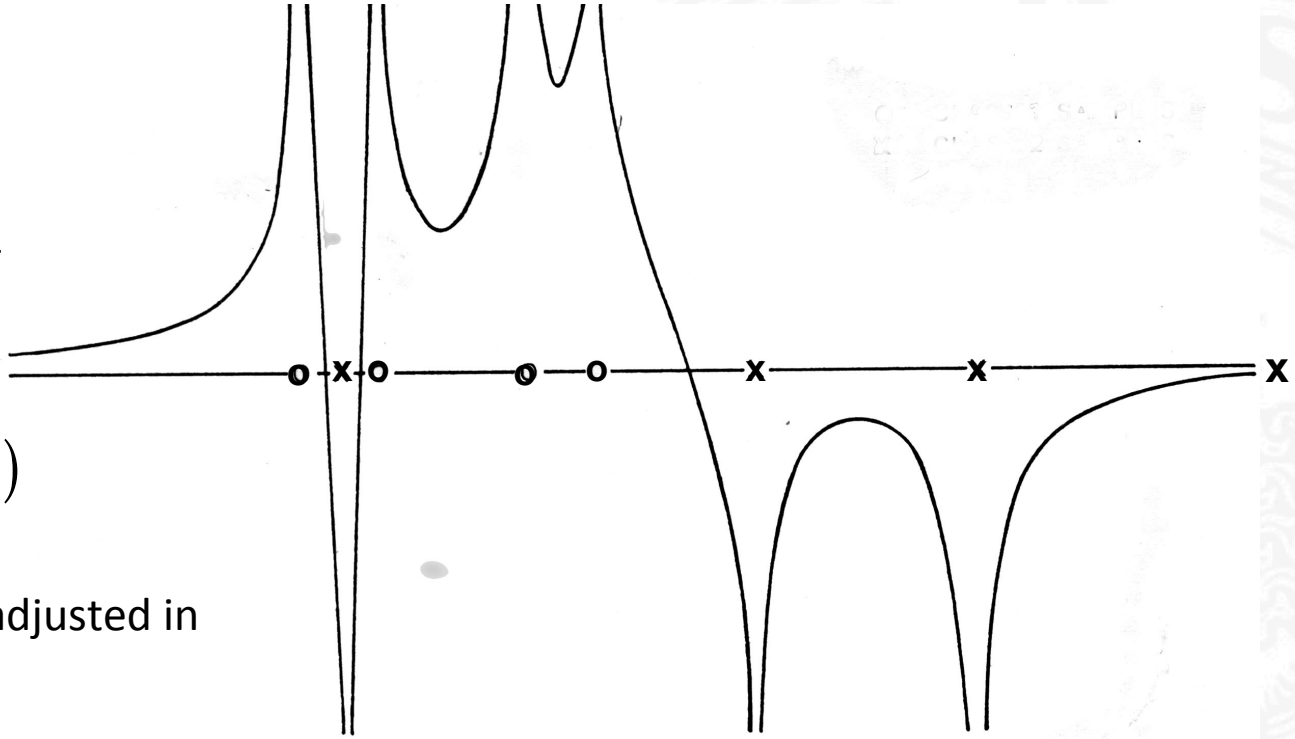
potential function

$$K(\vec{x}, \vec{x}_i) = \frac{1}{\|\vec{x} - \vec{x}_i\|^2}$$

discriminant function

$$g(\vec{x}) = \sum_i q_i K(\vec{x}, \vec{x}_i)$$

"charges" $q_i$ may be adjusted in learning procedure



# Construction of Discriminant Functions Based on Potential Functions

**Different choices for potential functions are possible, for example:**

$$K(\vec{x}, \vec{x}_k) = \frac{\sigma^2}{\sigma^2 + \left\|\vec{x} - \vec{x}_k\right\|^2}$$

$$K(\vec{x}, \vec{x}_k) = e^{-\frac{1}{2\sigma^2}\left\|\vec{x} - \vec{x}_k\right\|^2}$$

**Potential functions must be tuned to provide the right kind of interpolation between samples!**

**Iterative construction:**

$$g'(\vec{x}) = \begin{cases} g(\vec{x}) + K(\vec{x}, \vec{x}_k) & \text{if } \vec{x}_k \text{ is of class 1 and } g(\vec{x}_k) \le 0 \\ g(\vec{x}) - K(\vec{x}, \vec{x}_k) & \text{if } \vec{x}_k \text{ is of class 2 and } g(\vec{x}_k) \ge 0 \\ g(\vec{x}) & \textit{otherwise} \end{cases}$$