DIGITIZATION OF NON-REGULAR SHAPES

Peer Stelldinger

Cognitive Systems Group, University of Hamburg,
Vogt-Koelln-Str. 30, D-22527 Hamburg, Germany
stelldinger@informatik.uni-hamburg.de

Abstract Only the very restricted class of r-regular shapes is proven not to change topology during digitization. Such shapes have a limited boundary curvature and cannot have corners. In this paper it is shown, how a much wider class of shapes, for which the morphological open-close and the close-open-operator with an r-disc lead to the same result, can be digitized correctly in a topological sense by using an additional repairing step. It is also shown that this class is very general and includes several commonly used shape descriptions. The repairing step is easy to compute and does not change as much pixels as a preprocessing regularization step. The results are applicable for arbitrary, even irregular, sampling grids.

Keywords: shape, digitization, repairing, topology, reconstruction, irregular grid

Introduction

The processing of images by a computer requires their prior digitization. But as Serra already stated in 1980 [5], "To digitize is not as easy as it looks." Shapes can be regarded as binary images and the simplest model for digitization is to take the image information only at some sampling points and to set the associated pixels to these values. Unfortunately even for this simple digitization model only a very restricted class of binary images is proven to preserve topological characteristics during digitization: Serra proved that the homotopy tree (i.e. the inclusion properties of foreground- and background components) of r-regular images (see Definition 1) does not change under digitization with a hexagonal grid of certain density [5]. Similarily Pavlidis showed that such images can be digitized with square grids of certain density without changing topology [4]. Latecki [3] also referred to this class of shapes. Recently the author proved together with Köthe that r-regular sets are not only sufficient but also necessary to be digitized topologically correctly with *any* sampling grid of a certain density [6]. But most shapes are not r-regular, e.g. have corners. To solve this problem Pavlidis said "Indeed suppose that we have a class of objects whose contours contain corners. We may choose a radius of curvature

r and replace each corner by a circular arc with radius r" [4]. This approach to make shapes r-regular has two problems: (1) Pavlidis gives no algorithm how to do it exactly. He also does not say, for which shapes it is possible without changing the topology of the set. (2) It is a preprocessing step and thus cannot be computed by a computer, which only gets the digitized information. The aim of this paper is to solve both problems. After a short introduction in the definitions of r-regular images, sampling and reconstruction (section 1), the class of r-halfregular sets is defined in section 2, whose elements can be converted into r-regular sets by using a very simple morphological preprocessing step. In order to solve the second problem, it is shown how these shapes can be digitized topologically correctly by using a postprocessing algorithm instead of the preprocessing. These results are applicable for digitization with any type of sampling grid – only a certain density is needed. In section 3 it is shown that the concept of r-halfregular shapes includes several other shape descriptions. Finally in section 4 the postprocessing step is even more simplified in case of certain sampling grids. For square grids it simply means to delete all components and to fill all holes, which do not contain a 2x2 square of pixels. This is remarkably similar to the results of Giraldo et al. [2], who proved that finite polyhedra can be digitized with intersection digitization without changing their homotopy properties by filling all holes, which do not contain a 2x2 square. Unfortunately their approach was not applicable to other sets and was restricted to another digitization model.

1. Regular Images, Sampling and Reconstruction

At first some basic notations are given: The Euclidean distance between two points x and y is noted as $d(x, y)$ and the Hausdorff distance between two sets is the maximal distance between one point of one set and the nearest point of the other. The Complement of a set A will be noted as A^c. The boundary ∂A is the set of all common accumulation points of A and A^c. A set A is open, if it does not intersect its boundary and it is closed if it contains the boundary, $A^0 := A \setminus \partial A$, $\overline{A} := A \cup \partial A$. $\mathcal{B}_r(c) := \{x \in \mathbb{R}^2 | d(x, c) \leq r\}$ and $\mathcal{B}^0_r(c) := (\mathcal{B}_r(c))^0$ denote the closed and the open disc of radius r and center c. If $c = (0, 0)$, write $\mathcal{B}_r$ and $\mathcal{B}^0_r$. The r-dilation $A \oplus \mathcal{B}^0_r$ of a set A is the union of all open r-discs with center in A and the r-erosion $A \ominus \mathcal{B}^0_r$ is the union of all center points of open r-discs lying inside of A. The morphological opening with an open r-disc is defined as $A \circ \mathcal{B}^0_r := (A \ominus \mathcal{B}^0_r) \oplus \mathcal{B}^0_r$ and the respective closing as $A \bullet \mathcal{B}^0_r := (A \oplus \mathcal{B}^0_r) \ominus \mathcal{B}^0_r$. The concept of r-regular images was introduced independently by Serra [5] and Pavlidis [4]. These sets are extremely well behaved – they are smooth, round and do not have any cusps (e.g. see Fig. 2B). Furthermore r-regular sets are invariant under morphological opening and closing, as already stated by Serra [5].

DEFINITION 1 *A set $A \subset \mathbb{R}^2$ is called* r-regular *if for each boundary point of A it is possible to find two osculating open discs of radius r, one lying entirely in A and the other lying entirely in A^c.*

Each outside or inside osculating disc at some boundary point x of a set A defines a tangent through x, which is unique if there exists both an outside and an inside osculating disc. The definitions of r-erosion and r-dilation imply that the boundary of a set does not change under opening or closing with an r-disc iff it is r-regular. In order to compare analog with digital images, a definition of the processes of sampling and reconstruction is needed. The most obvious approach for sampling is to restrict the domain of the image function to a set of sampling points, called sampling grid. In most approaches only special grids like square or hexagonal ones are taken into account [3][4, 5]. A more general approach only needs a grid to be a countable subset of $\mathbb{R}^2$, with the sampling points being not too sparse or too dense anywhere [6]. There the pixel shapes are introduced as Voronoi regions. Together with Köthe the author proved a sampling theorem, saying that a closed r-regular image is $\mathbb{R}^2$-homeomorphic to its reconstruction with an r'-grid if only $r' < r$ [6]. Two sets being $\mathbb{R}^2$-homeomorphic means that there exists a homeomorphism from $\mathbb{R}^2$ to $\mathbb{R}^2$, which maps the sets onto each other.

DEFINITION 2 *A countable set $S \subset \mathbb{R}^2$ of* sampling points, *where the Euclidean distance from each point $x \in \mathbb{R}^2$ to the nearest sampling point is at most $r \in \mathbb{R}$, is called an* r-grid *if $S \cap A$ is finite for any bounded set $A \in \mathbb{R}^2$. The* pixel $\mathrm{Pixel}_S(s)$ *of a sampling point s is its Voronoi region, i.e. the set of all points lying at least as near to this point than to any other sampling point. The union of the pixels with sampling points lying in A is the* reconstruction of A w.r.t. S: $\hat{A} := \bigcup_{s \in S \cap A} \mathrm{Pixel}_S(s)$. *Two pixels are* adjacent *if they share an edge. Two pixels of $\hat{A}$ are* connected *if there exists a chain of adjacent pixels in $\hat{A}$ between them. A* component *of $\hat{A}$ is a maximal set of connected pixels.*

2. Digitization of Halfregular Sets

Most shapes are not r-regular for any r. So if one wants to apply the above mentioned sampling theorem one at first has to construct an r-regular version of the shape before sampling, as suggested by Pavlidis. The question is how to define such a preprocessing step. Obviously a set, which is the result of an r-opening of another set, has an inside osculating open r-disc at any boundary point. Equivalently the r-closing of any set has an outside osculating open r-disc at any boundary point. So the straight forward idea is to combine these two operators. Unfortunately the r-closing of an r-open set does not need to be r-open anymore and as a consequence the open-close and the close-open operator with the same structuring element can have totally different results. The really interesting case is when this does not happen:

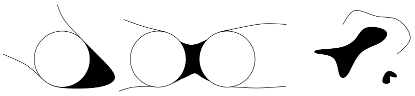

Figure 1. The areas which change during regularization can be classified info three types: r-tips (left), r-waists (center) and r-spots (right).

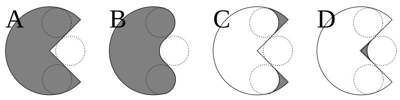

Figure 2. From left to right: An r-halfregular set A, its r-regularization B, and the changes of the fore- and background due to regularization C and D.

DEFINITION 3 *A set A is called* r-regularizable *if the open-close and the close-open-operator of radius r lead to the same (except of the boundary),* $\overline{(A \circ \mathcal{B}_r^0) \bullet \mathcal{B}_r^0 = (A \bullet \mathcal{B}_r^0) \circ \mathcal{B}_r^0}$. *The* r-regularization of A *is* $(A \circ \mathcal{B}_r^0) \bullet \mathcal{B}_r^0$.

LEMMA 4 *The r-regularization of an r-regularizable set A is r-regular.*

PROOF Since opening and closing are idempotent, $(A \bullet \mathcal{B}_r^0) \circ \mathcal{B}_r^0$ is open and $(A \circ \mathcal{B}_r^0) \bullet \mathcal{B}_r^0)$ is closed w.r.t. $\mathcal{B}_r^0$. This implies r-regularity of A. □

Note, that a shape and its regularization do not need to be $\mathbb{R}^2$-homeomorphic, since the topology can be totally changed during the regularization step. The changes can be classified into waists, tips and spots (see Fig. 1). The waists cause the biggest problems, because even big and thus important components can change their topology under regularization if they have waists. So if one wants to regularize a set it should have no waists.

DEFINITION 5 *For some set A let A' be a component of $(A \setminus (A \circ \mathcal{B}_r^0))^0$. Further let n be the number of open r-discs lying in A and touching A'. These discs are called* bounding discs *of A'. If n is zero, A' is called* r-spot *of A. If n is equal to 1, A' is called* r-tip *of A and if n is greater than 1, it is called* r-waist *of A (see Fig. 1). A set A is called* r-halfregular *if for each boundary point there exists an open inside or an open outside osculating disc of radius r, completely lying inside, respectively outside of A, and if neither A nor A^c has an s-waist for any $s \leq r$.*

Obviously an r-halfregular set is also s-halfregular for any $s < r$. For the rest of this section let A be an r-halfregular set with $r > 0$. Further let $B := (A \circ \mathcal{B}_r^0) \bullet \mathcal{B}_r^0$ be its r-regularization. $C := A \setminus (A \circ \mathcal{B}_r^0)$ shall be the difference between A and its opening with $\mathcal{B}_r^0$ and $D := (A \bullet \mathcal{B}_r^0) \setminus A$ the difference between A and its closing with $\mathcal{B}_r^0$ (see Fig. 2). The components of C and D are the r-spots, r-tips and r-waists, which change during the preprocessing regularization step.

LEMMA 6 *For each boundary point of C or of D there exists an outside osculating open r-disc and no component of $C \oplus \mathcal{B}_r^0$ or of $D \oplus \mathcal{B}_r^0$ contains an open disc of radius $2r$ as subset.*

PROOF C can have no open disc of radius r as subset, because due to the definition of C the center of such a disc is not in C. Now let x be a boundary point of C. Then x is either also boundary point of A or of $A \circ \mathcal{B}_r^0$. In the first case there exists an osculating open r-disc lying completely outside of C since A is r-halfregular and C cannot include an inside osculating r-disc. In the second case there also exists an outside osculating disc for C, since $A \circ \mathcal{B}_r^0$ is open w.r.t. $\mathcal{B}_r^0$. Thus $\overline{C}$ is closed w.r.t. $\mathcal{B}_r^0$ and $C \oplus \mathcal{B}_r^0$ does not contain any open disc of radius $2r$ as subset. The proof for D is analog. $\square$

LEMMA 7 *Let A be an r-halfregular set. Then every boundary point $y \in \partial A$ is also boundary point of $A \bullet \mathcal{B}_r^0$ or $A \circ \mathcal{B}_r^0$ and A is r-regularizable.*

PROOF Let y be some boundary point of A. If there exists an outside [inside] osculating r-disc, then y remains boundary point after r-closing [r-opening]. Now suppose A is not r-regularizable. Then $(A \circ \mathcal{B}_r^0) \bullet \mathcal{B}_r^0 \neq \overline{(A \bullet \mathcal{B}_r^0) \circ \mathcal{B}_r^0}$ and there either exists a point $x \in (A \circ \mathcal{B}_r^0) \bullet \mathcal{B}_r^0$, which is not in $\overline{(A \bullet \mathcal{B}_r^0) \circ \mathcal{B}_r^0}$ or there exists a point $x \in \overline{(A \bullet \mathcal{B}_r^0) \circ \mathcal{B}_r^0}$, which is not in $(A \circ \mathcal{B}_r^0) \bullet \mathcal{B}_r^0$. Such an x cannot lie inside or on the boundary of an r-disc being subset of A, because then x would be element of $\overline{A \circ \mathcal{B}_r^0}$ and thus $x \in (A \circ \mathcal{B}_r^0) \bullet \mathcal{B}_r^0$, and – since closing is extensive and opening is increasing – $x \in \overline{(A \bullet \mathcal{B}_r^0) \circ \mathcal{B}_r^0}$ is true. If x lies inside or on the boundary of an r-disc in A^c, it cannot be in $(A \circ \mathcal{B}_r^0) \bullet \mathcal{B}_r^0$ or $\overline{(A \bullet \mathcal{B}_r^0) \circ \mathcal{B}_r^0}$ for analog reasons. Now suppose, x is in A, but not inside or on the boundary of some r-disc in A, thus $x \in C^0$. C has an outside osculating open r-disc at any boundary point y due to Lemma 6. Now let y be the boundary point of C being nearest to x. Then any y tangent has to be orthogonal to $\overline{xy}$. Thus there exists a unique tangent and also a unique outside osculating r-disc. Obviously the distance between x and y is smaller than r. Since y remains boundary point after r-closing of C, x cannot be in $\overline{(A \bullet \mathcal{B}_r^0) \circ \mathcal{B}_r^0}$. $x \oplus \mathcal{B}_r^0$ can only intersect one outside osculating open r-disc of C lying inside $A \circ \mathcal{B}_r^0$, because there exists at most one such disc for each component C' of C^0 due to the absense of r-waists. The center point of this disc is the only point of $A \ominus \mathcal{B}_r^0$ having a distance of at most r to some point lying in C' or in $\partial C'$. Thus x cannot be element of $(A \circ \mathcal{B}_r^0) \bullet \mathcal{B}_r^0$. Analogously any x in D^0 is element of both $(A \circ \mathcal{B}_r^0) \bullet \mathcal{B}_r^0$ and $\overline{(A \bullet \mathcal{B}_r^0) \circ \mathcal{B}_r^0}$. Thus any $x \in \mathbb{R}^2$ is element of $(A \circ \mathcal{B}_r^0) \bullet \mathcal{B}_r^0$ iff it is element of $\overline{(A \bullet \mathcal{B}_r^0) \circ \mathcal{B}_r^0}$. $\square$

As a consequence of Lemma 7 A can be constructed (except of its boundary) as $A^0 = (B \cup C)^0 \ominus D$. In the following the sampling theorem for halfregular sets is developed. Therefore one lemma needs to be proved before.

LEMMA 8 *No background [foreground] component in the reconstruction of $B \cup C$ [$B \setminus D$] w.r.t. an r'-grid, $r' < r$, is subset of $C \oplus \mathcal{B}_r^0$ [$D \oplus \mathcal{B}_r^0$].*

PROOF Let $c \in A^c$ be a background sampling point in $C \oplus \mathcal{B}_r^0$. Due to Lemma 6 there exists an open r-disc in C^c such that c lies in the disc. This disc can be chosen such that it lies either completely in B or completely outside of B. The center m of the disc is not in $C \oplus \mathcal{B}_r^0$. The halfline starting at c and going through m crosses $\partial\,\mathrm{Pixel}(c)$ at exactly one point c'. If $d(c,m) \leq d(c,c')$, m lies in $\mathrm{Pixel}(c)$ and thus the pixel is connected to the area outside of $C \oplus \mathcal{B}_r^0$, which implies that c cannot be part of a separate background component covered by $C \oplus \mathcal{B}_r^0$. If $d(c,m) > d(c,c')$, let g be the line defined by the edge of $\mathrm{Pixel}(c)$ going through c'. If there are two such lines (i.e. at a pixel corner), one is chosen arbitrarily.p he point c'' constructed by mirroring c on g is also a sampling point, and their pixels are adjacent. c'' lies on the circle of radius $d(d',c) = d(c',c'')$ with center c'. Among all points on this circle, c has the largest distance to m, and in particular $d(m,c'') < d(m,c)$. Thus, c'' lies outside of $C \oplus \mathcal{B}_r^0$, and is closer to m than c. By repeating this construction iteratively we obtain a chain of adjacent pixels whose sampling points successively get closer to m. Since $C \oplus \mathcal{B}_r^0$ contains only a finite number of sampling points, one such pixel will eventually not be covered by $C \oplus \mathcal{B}_r^0$. The constructed chain consists of pixels whose sampling points lie in a common r-disc outside of C. If this disc lies in B, they are not in the background, in contradiction to the supposition. Otherwise the pixels cannot be part of a separate background component in $C \oplus \mathcal{B}_r^0$. Since the chain is not infected by any sampling point lying in B they also cannot be part of a separate background component in the reconstruction of $B \cup C$ which is subset of $C \oplus \mathcal{B}_r^0$. Analogously there exists no foreground component in the reconstruction of $B \setminus D$, which is subset of $D \oplus \mathcal{B}_r^0$. □

THEOREM 9 *Let A be a closed r-halfregular set with no $3r$-spot in A or in A^c, let $\hat{A}$ be the reconstruction of A with an r'-grid, $r' < r$, and let $\hat{A}'$ be the result of filling [deleting] all components of $\hat{A}^c$ [$\hat{A}$], which do not contain an open $2r$-disc. Then $\hat{A}'$ is $\mathbb{R}^2$-homeomorphic to A and the number of different pixels from $\hat{A}$ to $\hat{A}'$ is as most as high as from $\hat{A}$ to the reconstruction of the r-regularization B (see Fig. 3).*

PROOF $\hat{A}$ is equal to the union of the reconstructions of B and $C = A \setminus (A \circ \mathcal{B}_r^0)$ minus the reconstruction of $D = (A \bullet \mathcal{B}_r^0) \setminus A$. Due to Lemmas 4 and 7 is B r-regular and thus $\mathbb{R}^2$-homeomorphic to its reconstruction (see [6]). The components of $\hat{C}$ are either also separated components in $\hat{A}$ or connected with some component of $\hat{B}$. Lemma 6 states that no component of $C \oplus \mathcal{B}_r^0$ contains an open disc of radius $2r$. Due to $r' < r$, $\hat{C}$ is a subset of $C \oplus \mathcal{B}_r^0$. It follows that no component of $\hat{C}$ can contain an open disc of radius $2r$. Analogously no component of $\hat{D}$ can contain an open disc of radius $2r$. Due to Lemma 8 there cannot exist any background component in the reconstruction of $B \cup C$, which is subset of $C \oplus \mathcal{B}_r^0$, and there cannot exist any foreground component

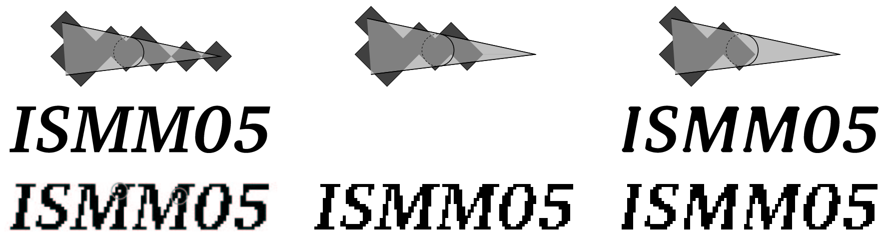

Figure 3. The straightforward digitization may cause topological errors at r-tips (left) in contrast to the use of the regularization step (right) and the even better repairing step (center).

in the reconstruction of $B \setminus D$, which is subset of $D \oplus \mathcal{B}_r^0$. This implies that any separate component of $\hat{C}$ or of $\hat{D}$ is surrounded by pixels belonging to components which do not vanish under dilation or erosion with an open $3r$-disc. This also implies that the resulting image is independent of the order of the filling and deleting of components, which are subsets of $C \oplus \mathcal{B}_r^0$ or of $D \oplus \mathcal{B}_r^0$. So by filling all components of $\hat{A}^c$ and deleting all components of $\hat{A}$, which do not contain an open disc of radius $2r$, any component caused by C and D is affected, which is not part of bigger components in $\hat{A}$ and $\hat{A}^c$, respectively. Any component of $\hat{B}$ [$\hat{B}^c$] is not deleted [filled], because it contains an open disc of radius $2r$ due to the fact that the corresponding component in B [B^c] contains a disc of radius $3r$. It follows that the components of $\hat{A}'$ and $\hat{B}$ differ only in a way, which does not affect the topology or the neighborhood relations. Thus $\hat{A}'$ is $\mathbb{R}^2$-homeomorphic to B. Moreover they differ only in pixels lying in C or D. Since $\hat{A}$ and $\hat{A}'$ differ in all these pixels, the number of pixels, which changes due to the postprocessing repairing step is at most as big as the number of pixels, which changes due to the preprocessing regularization step. Since B can be constructed from A by removing r-tips from A and A^c, which is an $\mathbb{R}^2$-homeomorphic operation, $\hat{A}'$ is also $\mathbb{R}^2$-homeomorphic to A. □

3. Examples for r-halfregular sets

In the last section it was shown, that digitization with repairing is sufficient to get a topologically correct digital version of some r-halfregular set, if each of the components of the set and its complement has a certain size (no $3r$-spots). Note that this is only a restriction to the sampling density and not a restriction to the class of correctly digitizable sets, since for each r-halfregular set there exists an $s \leq r$ such that the set has no $3s$-spots and is s-halfregular. Surprisingly the concept of r-halfregular sets is very general and several commonly used shape descriptions imply halfregularity. One example are polygonal shapes. The first part of Theorem 10 shows that any shape with polygonal boundary description is r-halfregular for some r. This is in particular interesting since any shape bounded by simple curves can be approximated by a polygonal shape, as shown by Bing[1]. Giraldo et al. already proved that such polygonal shapes can be reconstructed topologically correctly by using a simple repairing step, which is very similar to the one of this paper [2]. But their approach is restricted to polygonal shapes and square grids and uses intersection digitization. Intersection digitization can be simulated by the digitization model of this paper on r-dilated shapes. The second part of Theorem 10 states that any convex set is halfregular. Since convexity is stable under projection from 3D to 2D the shape of any image of a convex object, like a ball, a cylinder or a box, can be topologically correctly digitized by using the repairing step. This implies that also sets whose complement is convex, are halfregular. Since halfregularity bases only on local properties, sets are also halfregular, if they or their complements are convex in each local area of certain size, as can be seen in the third part of the theorem. These are only some examples for halfregular shapes. Another example are shapes which are bounded by spline curves, i.e. true type fonts. There it is not difficult to determine the minimal size of the spots and waists and the maximal curvature at non-corner points, which can be used to compute the minimal sampling density in order to digitize a text printout, such that topology is preserved (see Fig. 3).

THEOREM 10 *(a) Let A be a set, where the boundary components are polygonal Jordan curves. Further choose $r > 0$ such that the Hausdorff distance from each line segment to each non-adjacent corner point is at least $4r$ and the Hausdorff distance between each two non-adjacent line segments is at least $2r$. Then A is r-halfregular. (b) Each convex set is r-halfregular for any $r \in \mathbb{R}_+$. (c) Each set A, where for each boundary point $x \in \partial A$ either $A \cap \mathcal{B}^0_{2r}(x)$ or $A^c \cap \mathcal{B}^0_{2r}(x)$ is convex, is r-halfregular.*

PROOF (a) Since any waist has to be bounded by at least two non-adjacent line segments and since the distance between any two non-adjacent line segments is at least $2r$, there cannot exist any s-waist for $s \leq r$. Let a be an arbitrary boundary point of A and let $B = \mathcal{B}^0_{2r}(a)$ be the disc of radius $2r$ centered in a. a lies on some line segment L of the boundary, whose endpoints shall be called l_1 and l_2. If a is a corner point, choose either of the two adjacent line segments. Since the Hausdorff distance of non-adjacent line segments is at least $2r$, only line segments, which are adjacent to L can lie in B. W.l.o.g. let the distance from a to l_1 be at most as big as the distance from a to l_2. Then the line segment L' being adjacent to L by meeting in l_2 has a Hausdorff distance of at least $4r$ to l_1. If the angle between L and L' is at least $\pi/2$, L' cannot go through B. Otherwise suppose the foot of the perpendicular of L' going through a is in L'. Then due to the theorem on intersecting lines the Hausdorff distance from L' to a is $d(L', a) = d(a, l_2)/d(l_1, l_2) \cdot d(L', l_1) \geq \frac{1}{2} \cdot 4r$. If the foot of the perpendicular is not in L', L' is shorter than L. Then L' also cannot intersect B, because otherwise L would intersect such a disc B' with radius $2r$ and center in L'. Thus the only line segments of the boundary, which can intersect B, are L and the adjacent line segment meeting L in l_1. The straight line which covers L cuts B into two open halfs such that L' intersects at most one of these halfs. The other half disc contains an open disc of radius r osculating a. This disc does not intersect the boundary, which implies r-halfregularity. □

(b) A convex set can be described as the intersection of halfplanes. Any outside osculating r-disc of some halfplane is completely outside of this intersection. Thus for any boundary point there exists an outside osculating r-disc. Obviously convex sets cannot have waists, which implies r-halfregularity. □

(c) First suppose there exists an s-waist A' of A with $s \leq r$. Now let x be a boundary point of A which is also boundary point of A'. If $A^c \cap \mathcal{B}^0_{2r}(x)$ is convex, then it is subset of a half of the $2r$-disc and the other half contains an r-disc lying in A. Thus $A \cap \mathcal{B}^0_{2r}(x)$ has to be convex. Since this has to be true for any such boundary point, A must be convex in the area $A' \oplus \mathcal{B}^0_{2r}$ of radius $2r$ around the waist. This implies that the union of the waist and its bounding discs also has to be convex. But this cannot be since the convex hull of this union is morphologically open w.r.t. $\mathcal{B}^0_s$ in contrast to the set itself. Thus there cannot exist any s-waist in A and analogously in A^c. For each boundary point x there exists an outside osculating r-disc if $A \cap \mathcal{B}^0_{2r}(x)$ is convex and an inside osculating disc if $A^c \cap \mathcal{B}^0_{2r}(x)$ is convex. Thus A is r-halfregular. □

4. Discrete Repairing

Although the repairing process is very simple – you only have to find components which do not contain a disc of a certain size – an implementation into a discrete algorithm is not straightforward since subpixel positions of such discs have to be considered. In this section it is shown that there are even better ways to find such components. The idea is that for any regular r'-grid there are only finitely many patterns which cover an r-disc ($r = r' + \varepsilon$) such that each pixel intersects the disc, and some patterns include others. So if one has a set of patterns such that any possible pattern is superset of an element of the set, one has only to look for components which do not include any of these elements. The following theorem shows this in detail for square and hexagonal grids.

THEOREM 11 *Let A be a closed r-halfregular set with no $3r$-spot in A or in A^c, let $\hat{A}$ be the reconstruction of A with a square grid [hexagonal grid] which is an r'-grid, $r' < r$, and let $\hat{A}'$ be the result of filling/deleting all components of $\hat{A}^c$ and $\hat{A}$, which do not contain the highlighted configuration shown in Fig. 4(a) [which do not contain one of the highlighted configurations in Fig. 1(b)]. Then $\hat{A}'$ is $\mathbb{R}^2$-homeomorphic to A and the number of different pixels between $\hat{A}$ and $\hat{A}'$ and $\hat{A}$ is as most as high as between $\hat{A}$ and the reconstruction of the r-regularization B.*

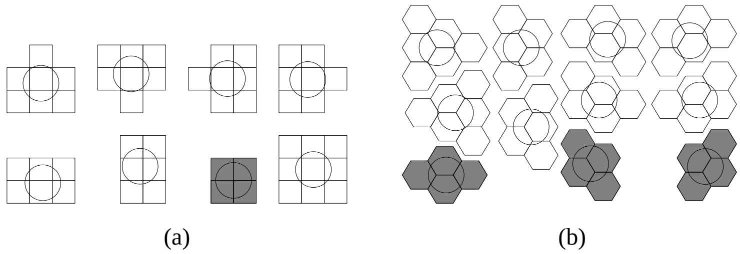

Figure 4. There is only a finite number of pixel configurations which cover an r-disc when using square (a) or hexagonal (b) grids and a minimal set of subconfigurations (highlighted).

PROOF If a component of the reconstruction contains an r-disc, then it also contains one of the configurations of Fig. 4(a) [4(b)]. The highlighted one [At least one of the highlighted configurations in Fig. 1(b)] is subset of any of these configurations. Otherwise if a component of the reconstruction contains such a highlighted configuration, it also contains an $(r' + \varepsilon)$-disc for a sufficiently small ε. This is all to show since A is also $(r' + \varepsilon)$-halfregular. □

5. Conclusions

The new class of r-halfregular shapes was firstly introduced and it was shown that this class can be digitized topologically correctly by using a simple post-processing step. The main result simply says that the digitization of an r-halfregular shape with an arbitrary sampling grid of sampling density $r' < r$ is topologically undistinguishable from the original shape after applying a post-processing step which simply removes all components, which do not exceed a certain size. This is much more general than the restriction to r-regular shapes, which was used in literature before. Is was also shown that the postprocessing step leads to better results than a morphological preprocessing step, which makes an r-halfregular shape r-regular. Further on it was proven that the class of r-halfregular shapes subsumes other shape classes like polygonal or convex shapes. Finally the postprocessing step was even more simplified in case of using regular sampling grids like square or hexagonal ones.

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