Flow Shapes

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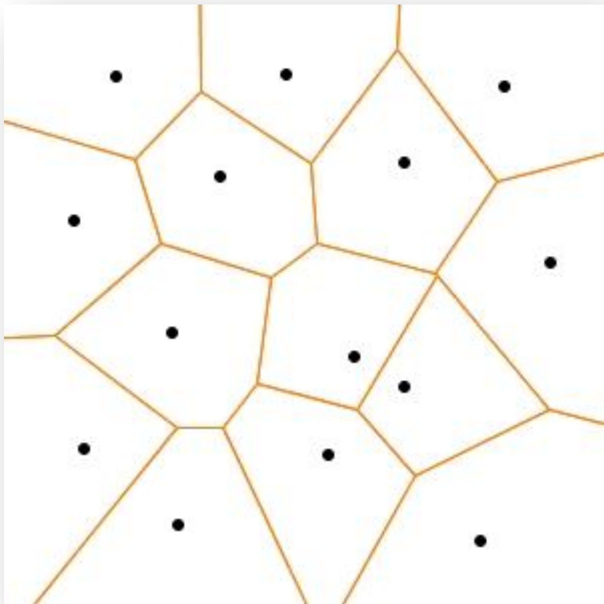
Seminar “Surface Reconstruction”
19.11.2008

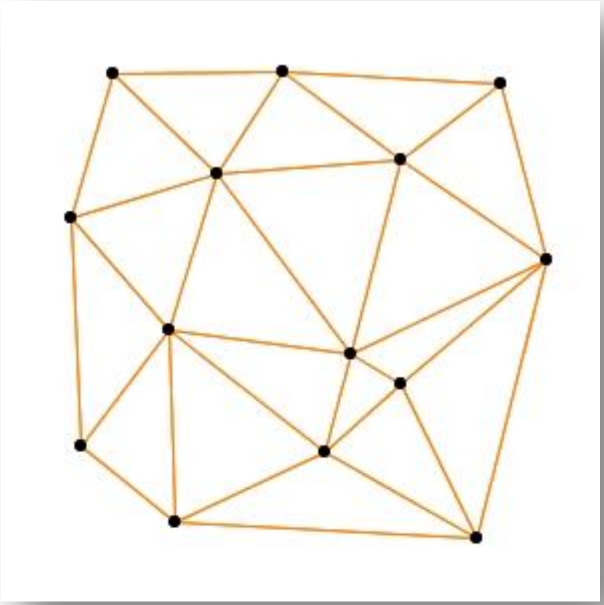
Contents

Voronoi diagram



- Voronoi objects:
  - Voronoi vertex
  - Voronoi edge
  - Voronoi face
  - Voronoi cell

Delaunay diagram



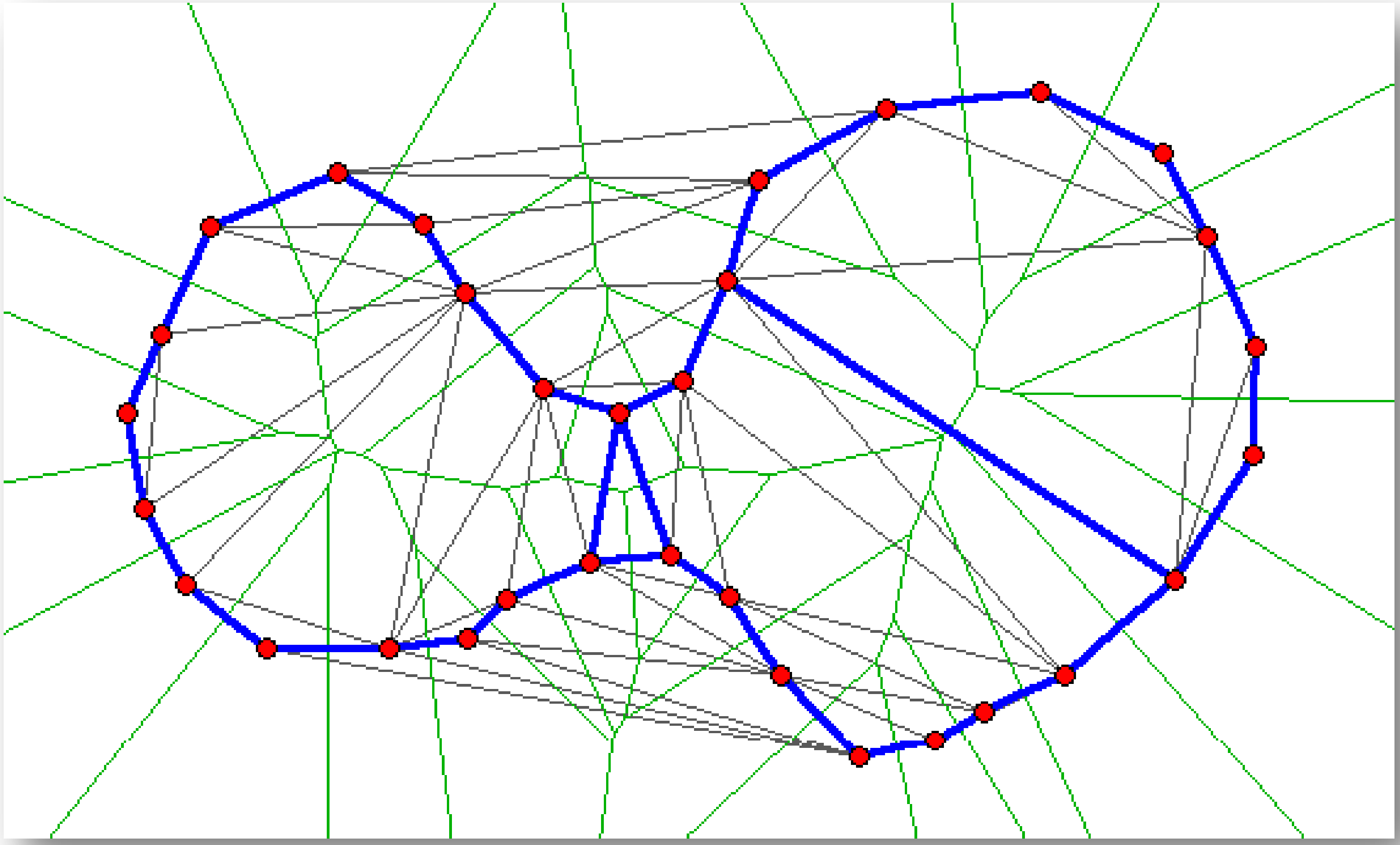
- Delaunay objects:
  - Delaunay vertex
  - Delaunay edge
  - Delaunay face
  - Delaunay cell

Gabriel graph

- Vertices are the points in P
- Edges are given by Delauney edges that intersect their dual Voronoi facet

- Always connected (contains the Euclidean minimum spanning tree)
- Facets and edges: their smallest circumscribing ball is empty of sample points

Gabriel graph (2)



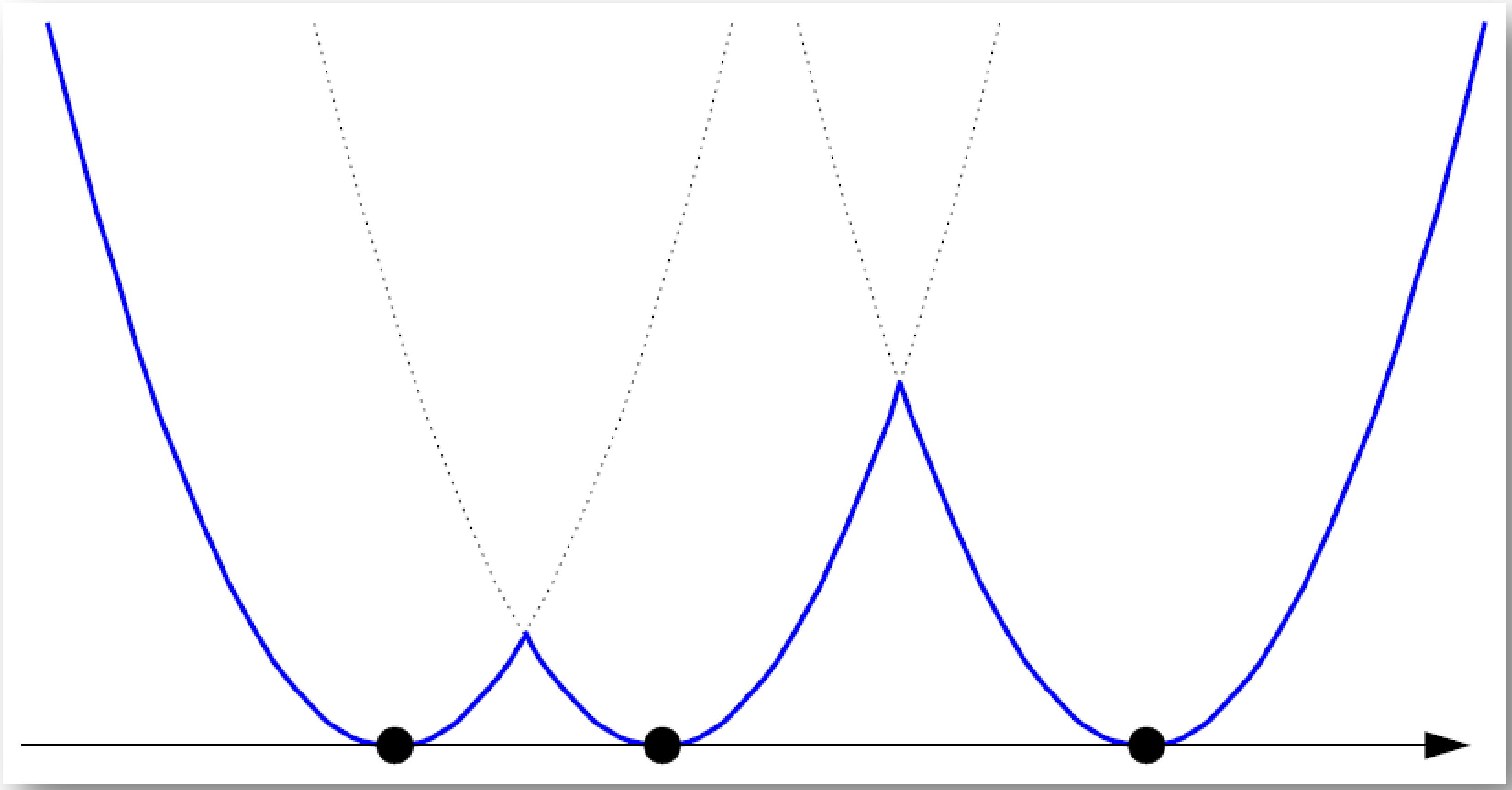
Distance function

- Assigns to every point in R^3 its least distance to any of the sample points

$$h(x) = \min\{|x - p|^2 : p \in P\}, \quad P \subset R^3$$

- Funtion *h* is continiuous
- Smooth everywhere besides at point which have the same distance from two or more points

Distance function (2)

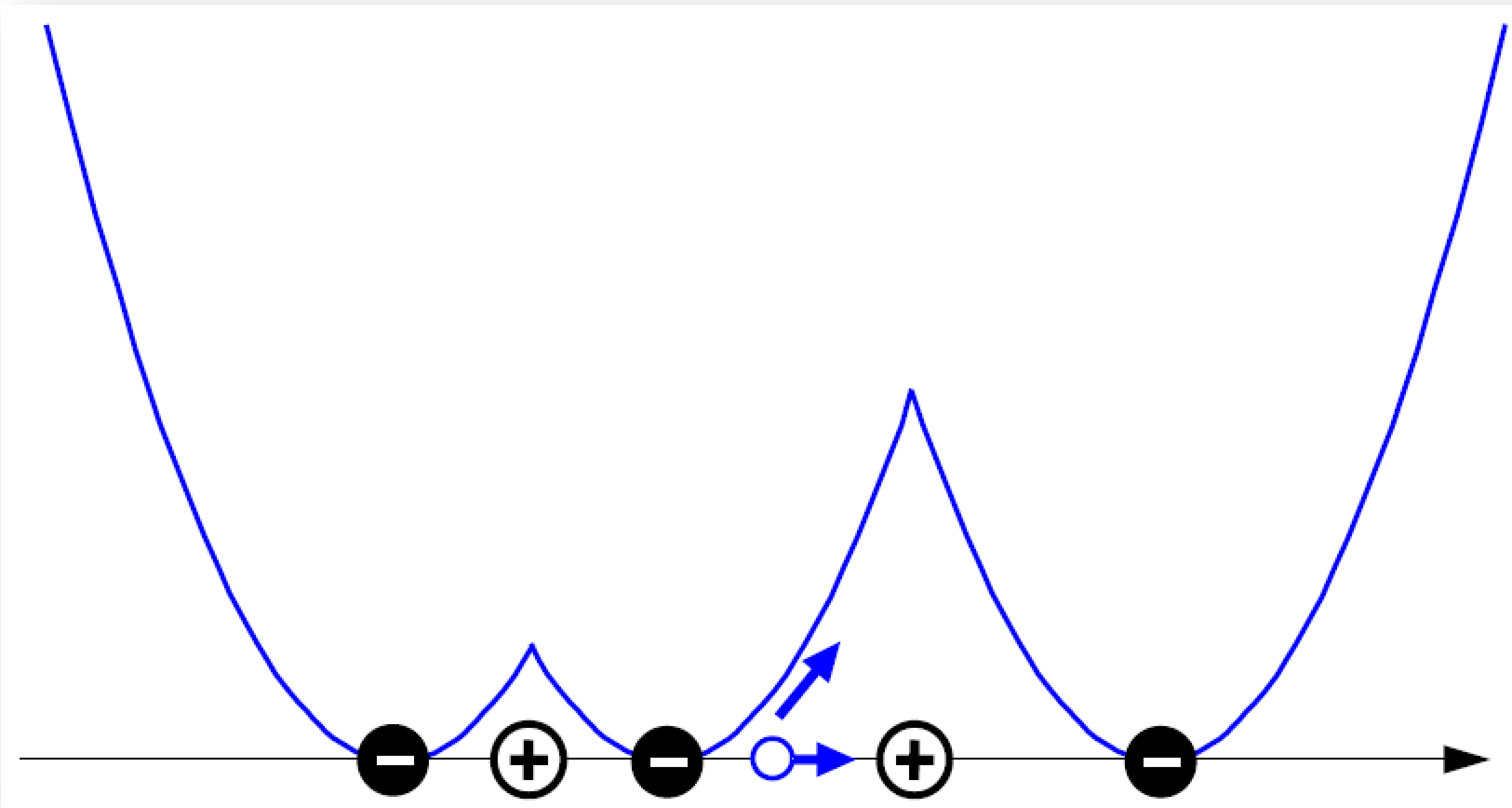


Critical points

- The critical points of the distance function h are the intersection points of Voronoi objects V and their dual Delaunay object σ. The index of a critical point is the dimension of σ.

- Points where a unique direction of steepest ascent of the distance function does not exist

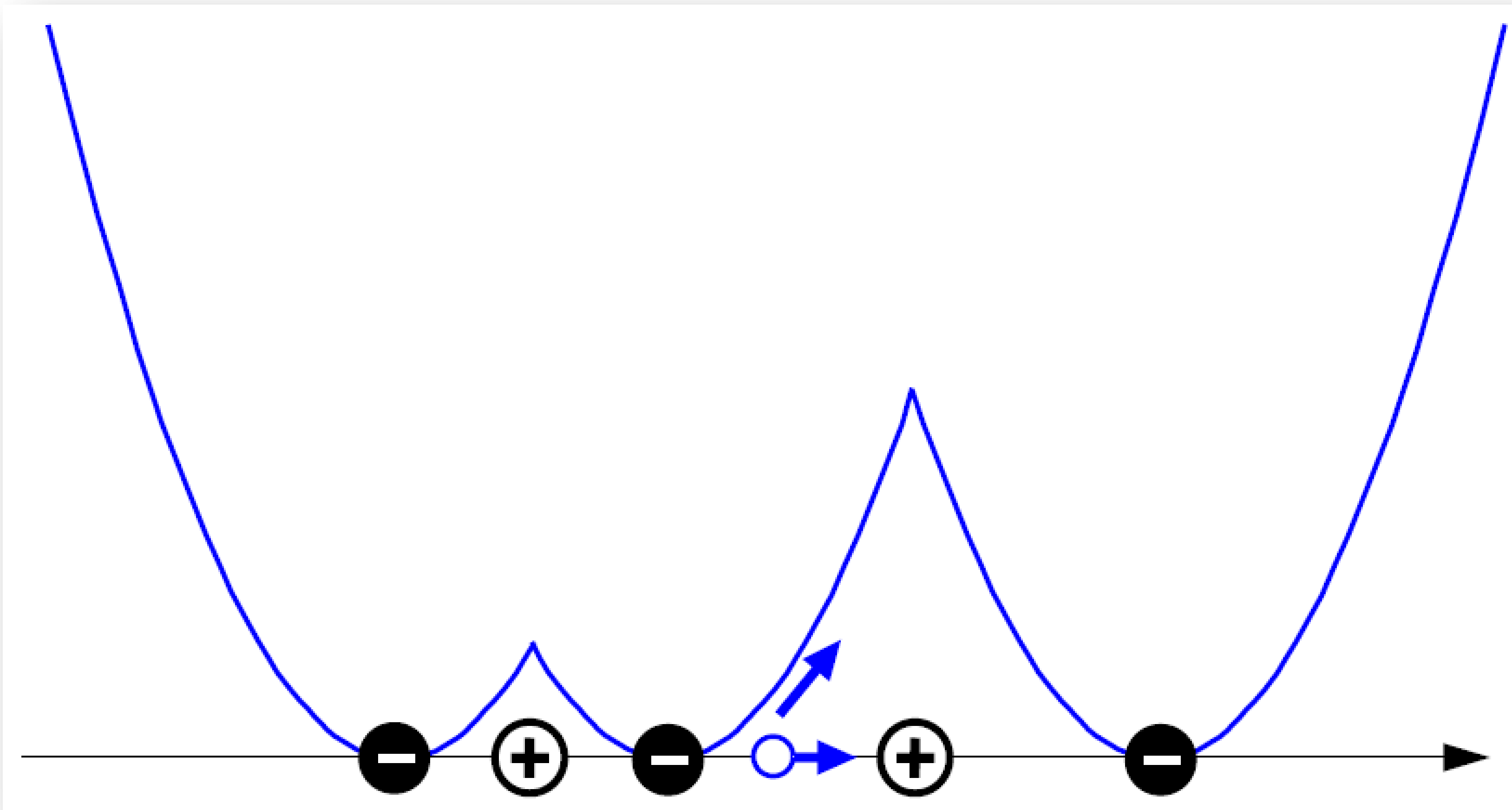
Critical points (2)



Critical points (3)

- Index-0
  - the sample points themselves
  - local minima of the distance function *h*.
- Index-1
  - intersection of Delaunay edges and their dual Voronoi facets
  - only Gabriel edges intersect their dual Voronoi facet
- Index-2
  - intersection of Delaunay facets and their dual Voronoi edges
  - not all Delaunay facets contain an index-2 saddle point
- Index-3
  - intersection of Delaunay cells and their dual Voronoi vertices.
  - local maxima for h.

Critical points (4)



Driver

- Given $x \in R^3$. Let V be the lowest dimensional Voronoi object in the Voronoi diagram of P that contains x and let σ be the dual Delauney objekt of V. The driver $d(x)$ of x is the point on σ closest to x.

- Direction of the steepest ascent of the distance function h:

$$v(x) = \frac{x - d(x)}{|x - d(x)|}$$

Induced flow ϕ

- For all critical points *x* we set:

$$\phi(t, x) = x\ ,\ t \in [0, \infty)$$

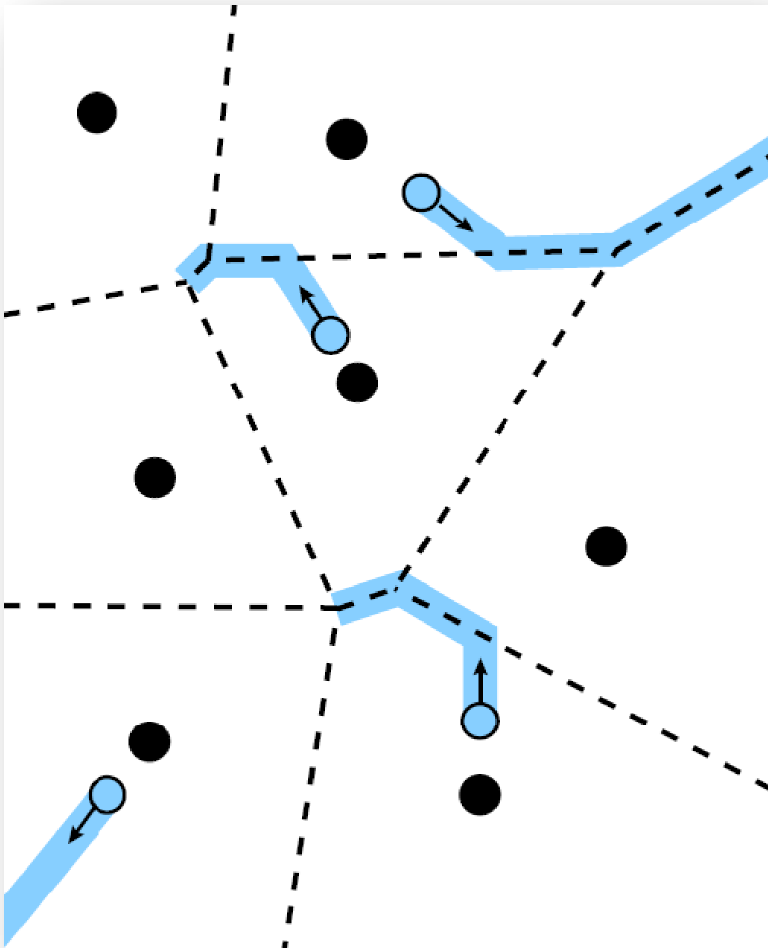
- Otherwise:
  - *R* is the ray originating at x and shooting in the direction of the steepest ascent *v*(*x*)
  - *z* be the first point on *R* whose driver is different from *d*(*x*)
    - such a z need not exist in R^3 if x is contained in an unbounded Voronoi object
    - in this case z be the point at infinity in the direction of R
  - We set:

$$\phi(t, x) = x + t \cdot v(x)\ ,\ t \in [0, |z - x|)$$

  - For $t \geq |z - x|$ the flow is given as follows:

$$\begin{aligned} \phi(t, x) &= \phi(t - |z - x| + |z - x|, x) \\ &= \phi(t - |z - x|, \phi(|z - x|, x)) \end{aligned}$$

Orbit ϕ_x

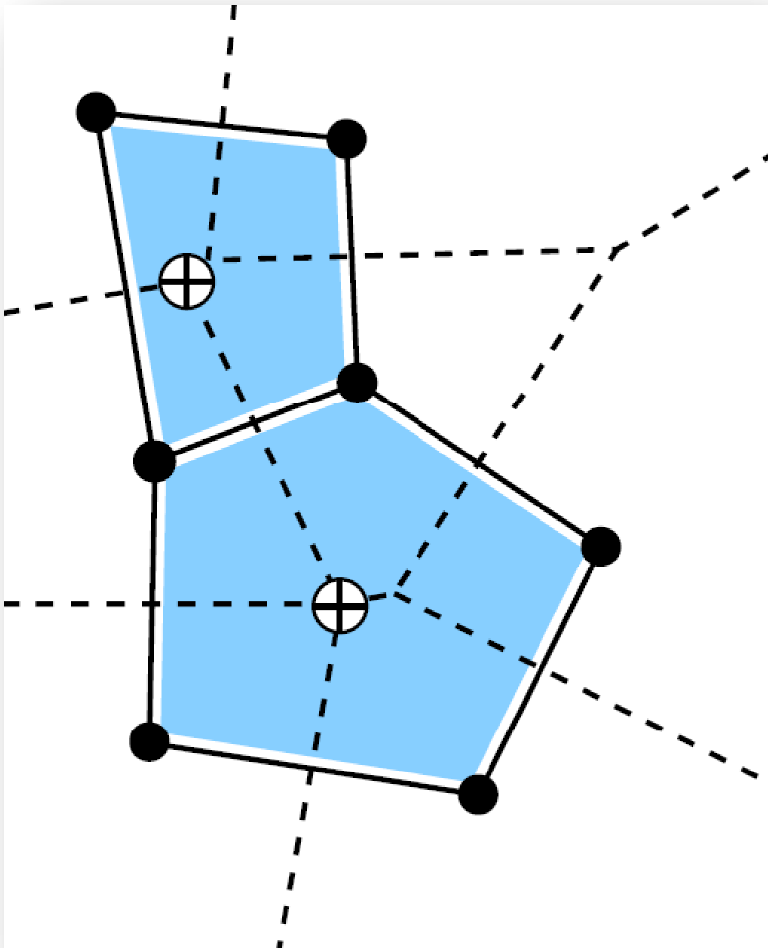


- Orbit of *x*:

  $\phi_x : [0,\infty) \rightarrow R^3 , t \rightarrow \phi(t, x)$

- Fixpoint of ϕ:
  - A point *x* if $\phi_x(t) = x$ for all $t \geq 0$
  - Fixpoints of ϕ are the critical points of $h(x)$

Stable manifolds



- The stable manifold $S(x)$ of a fixpoint $x \in \mathbb{R}^3$ is the set of all points that flow into x:

$$S(x) = \{y \in \mathbb{R}^3 : \lim_{t \to \infty} \phi_y(t) = x\}$$

Flow complex

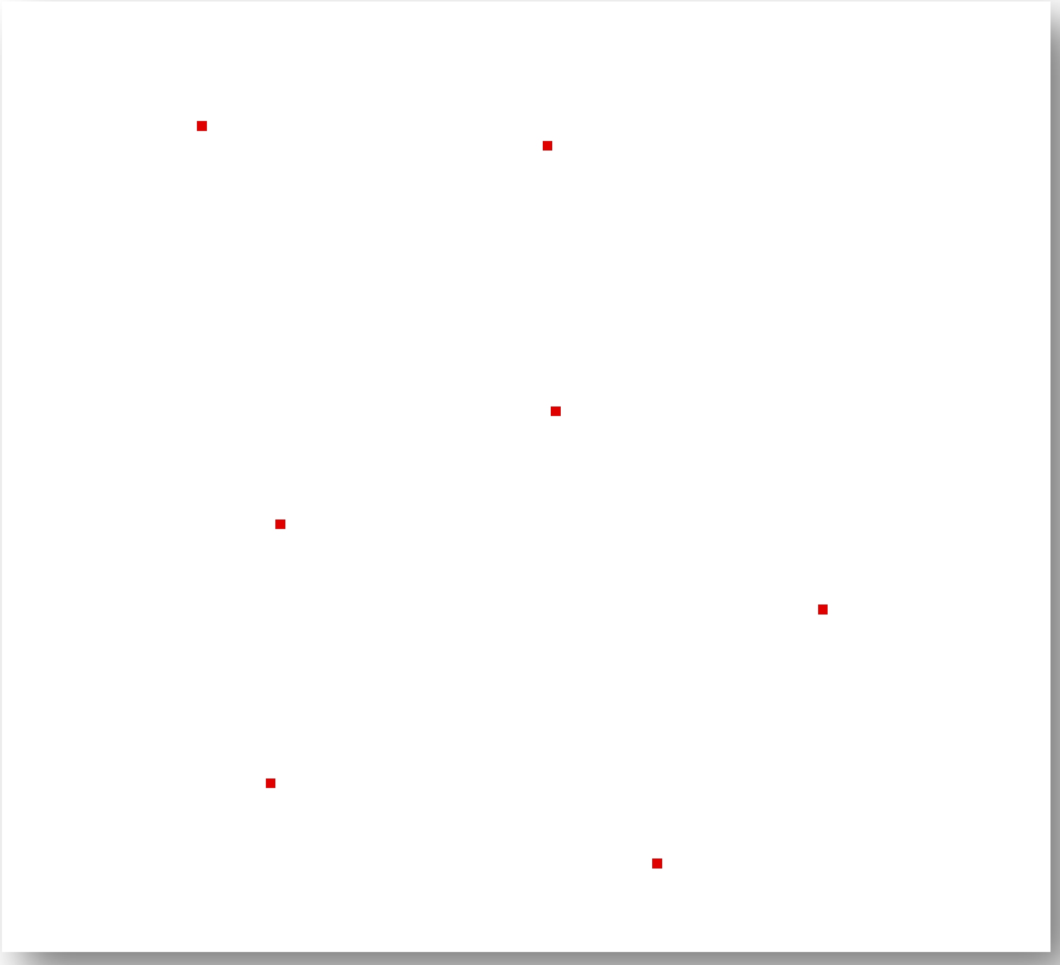
- $F^{\alpha}(P)$ is the collection of all stable manifolds of critical points x with $h(x) \leq \alpha$
- $F^{\alpha}(P) = P$ for $\alpha \leq 0$

- Flow shape is the underlying topological space of $F^{\alpha}(P)$

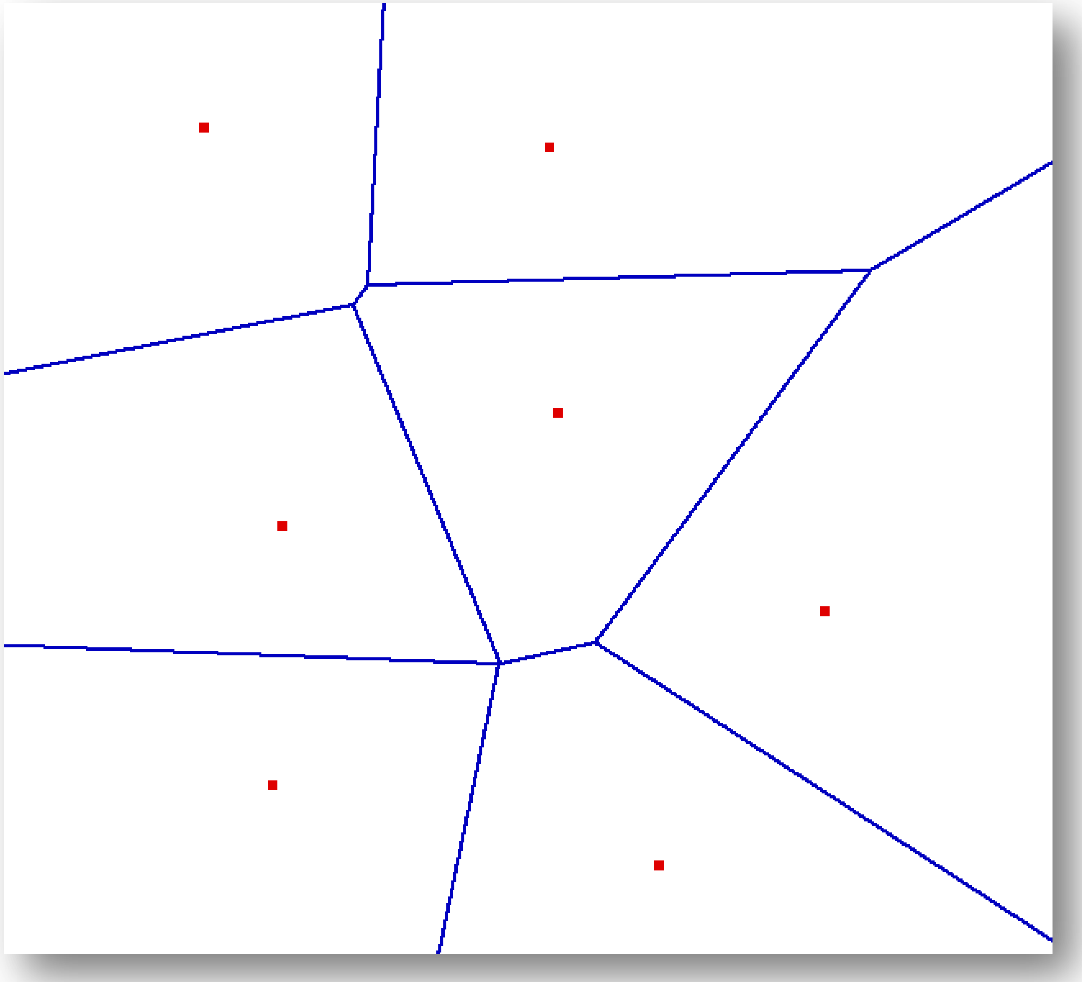
Sources

- Joachim Giesen, Matthias John : The flow complex: A data structure for geometric modeling (2003)
- Tamal K. Dey, Joachim Giesen, Matthias John: Alpha-Shapes and Flow Shapes are Homotopy Equivalent (2003)
- Bálint Miklós: Geometric Modelling with the Flow Complex (2005)

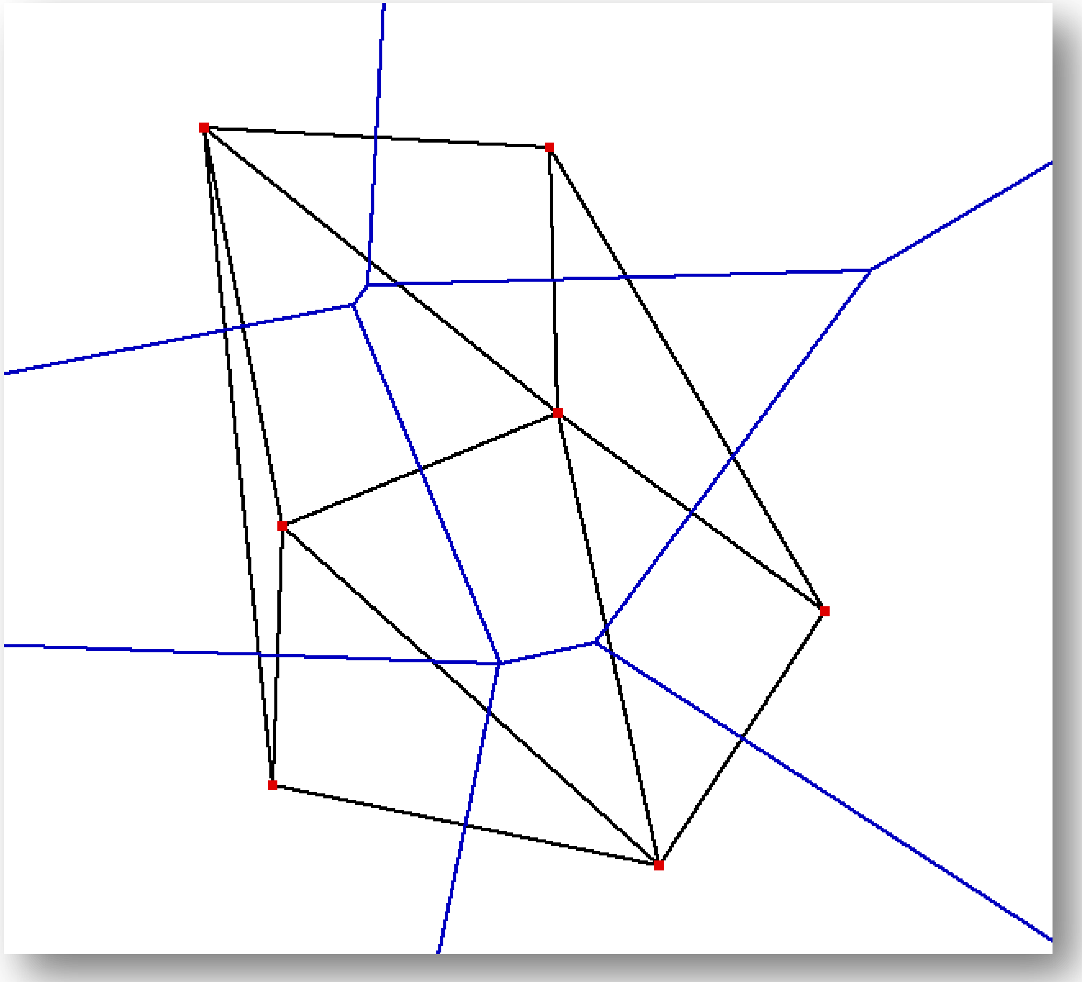
Sample points



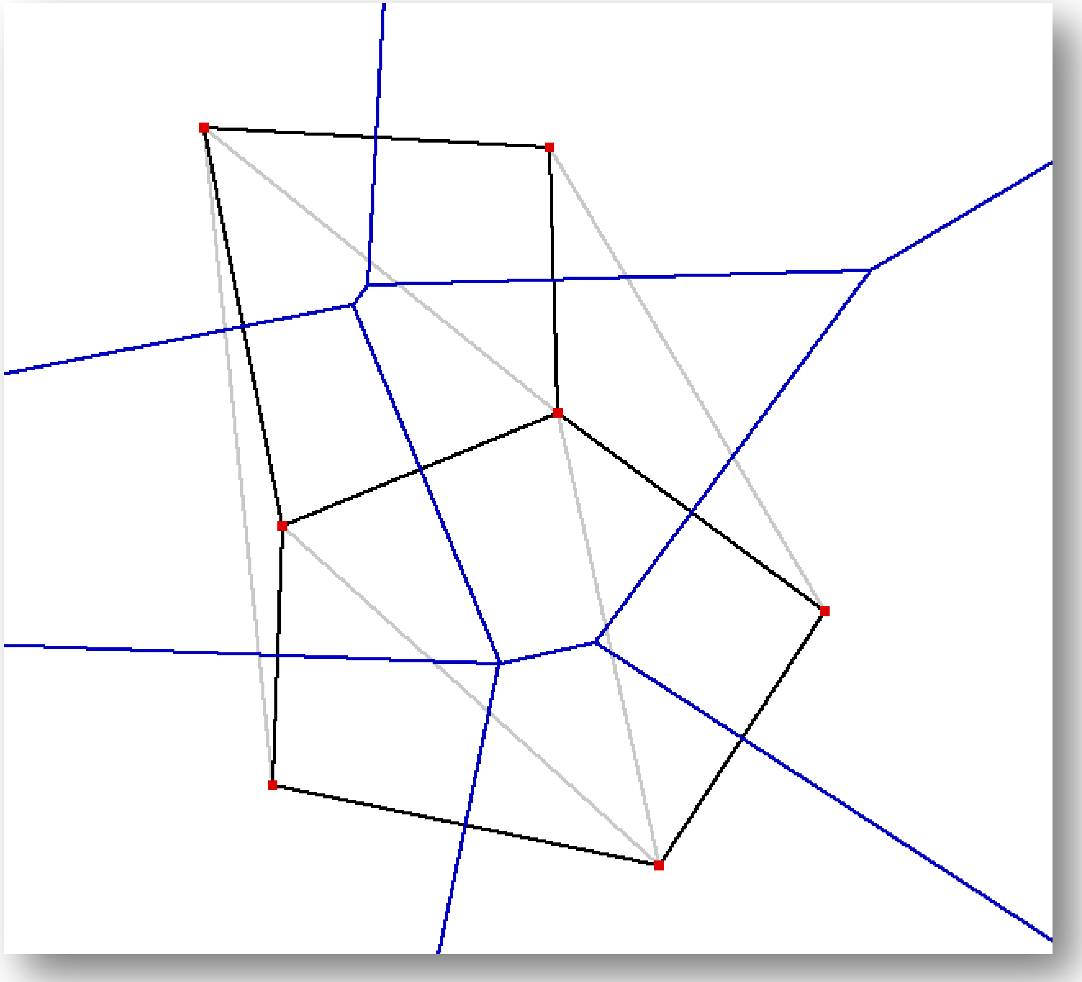
Voronoi diagram



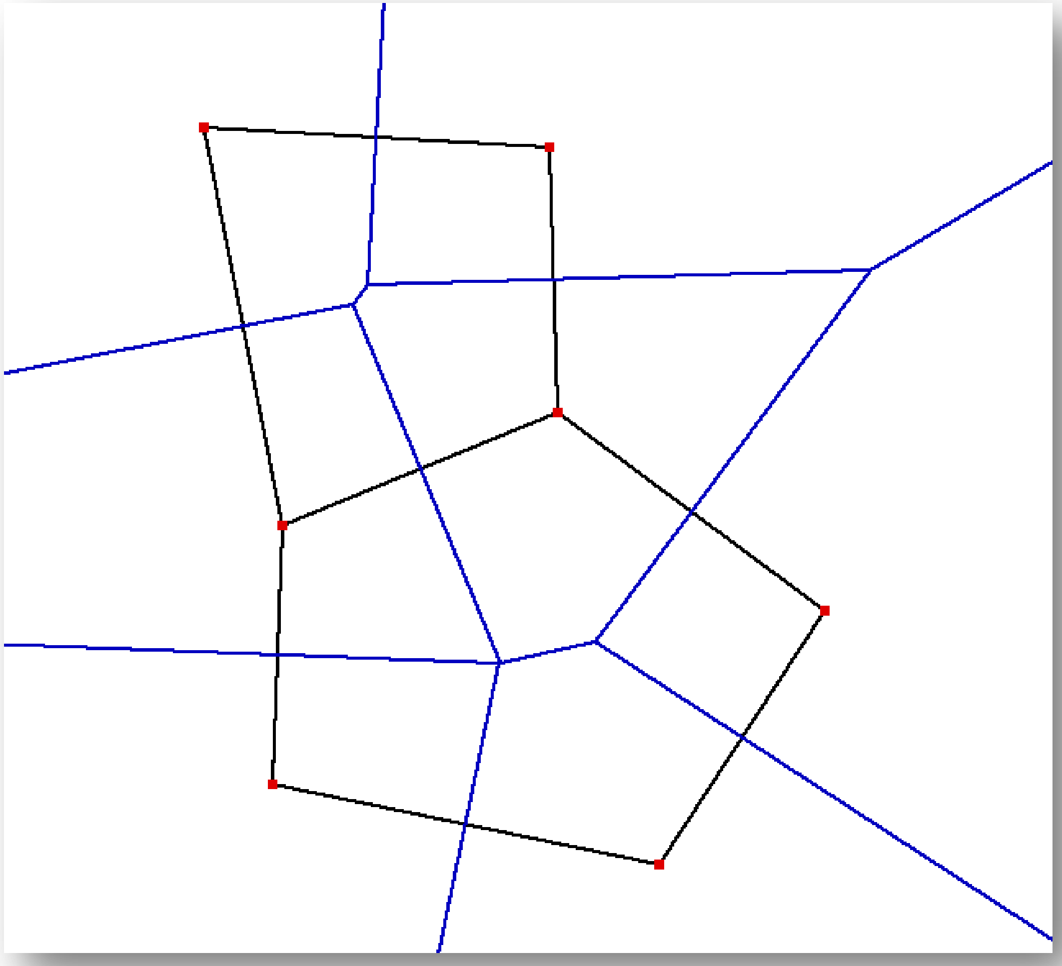
Voronoi diagram + Delaunay diagram



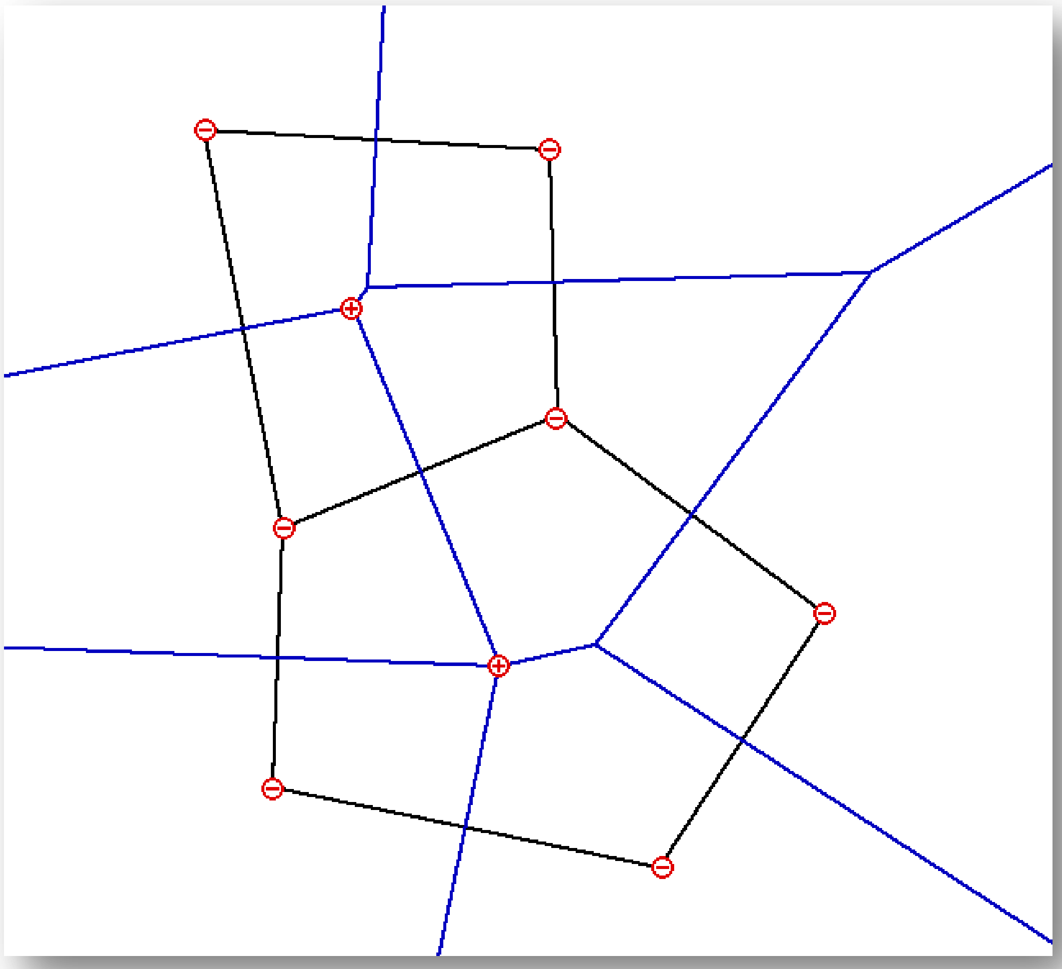
Voronoi diagram + Gabriel graph



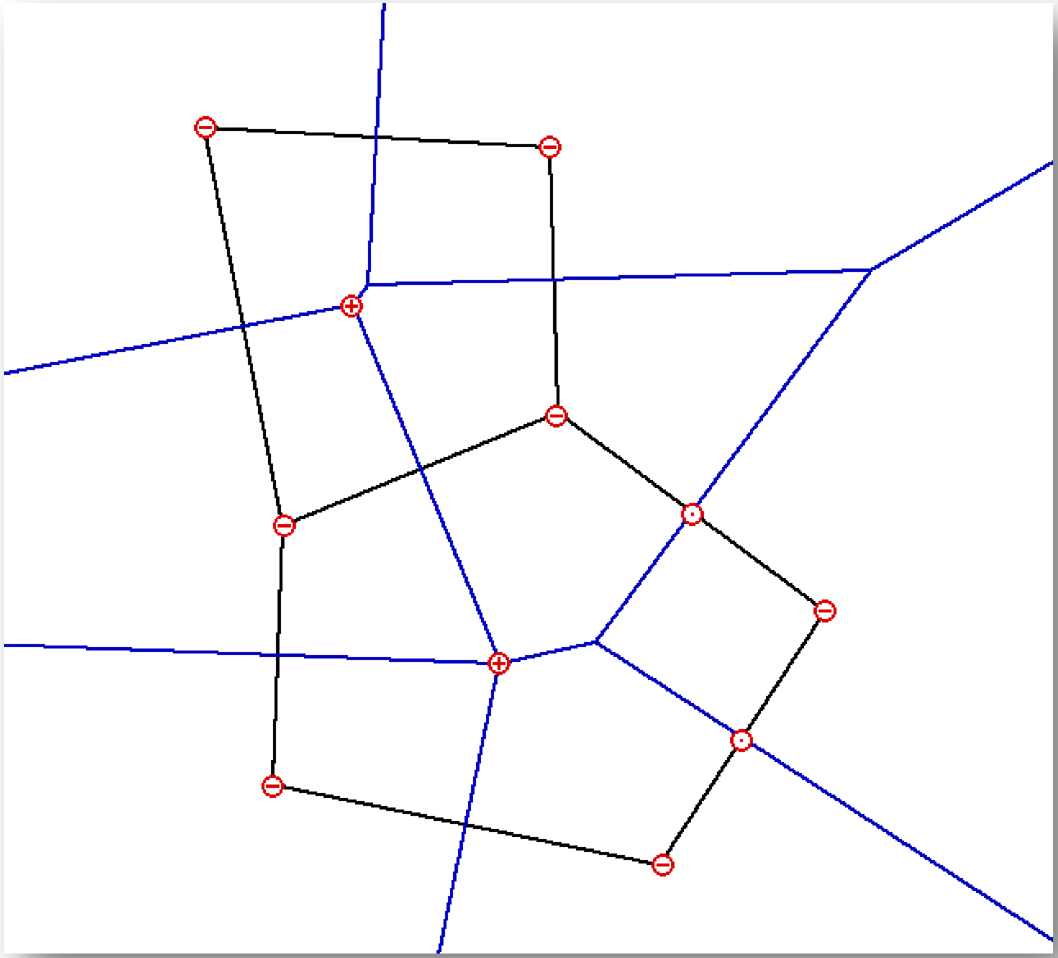
Voronoi diagram + Gabriel graph (2)



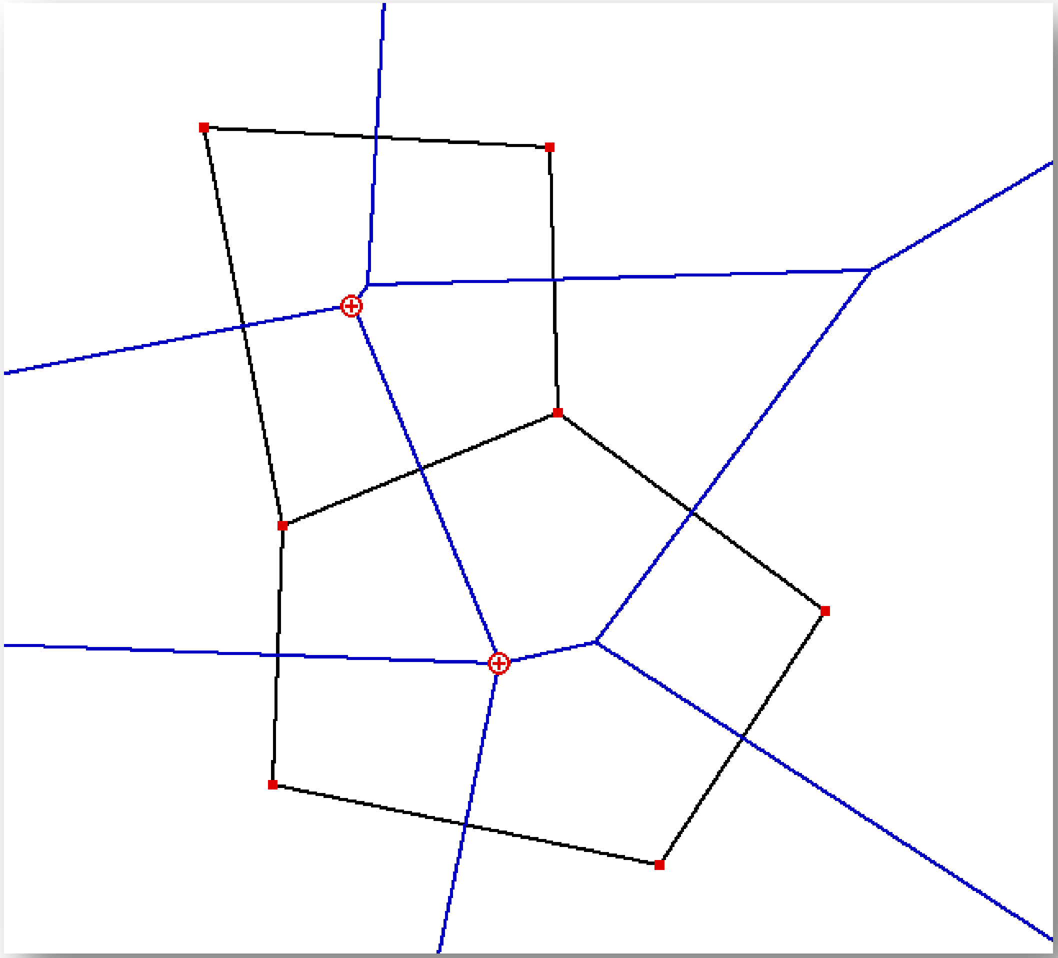
Minima + maxima of h(x)



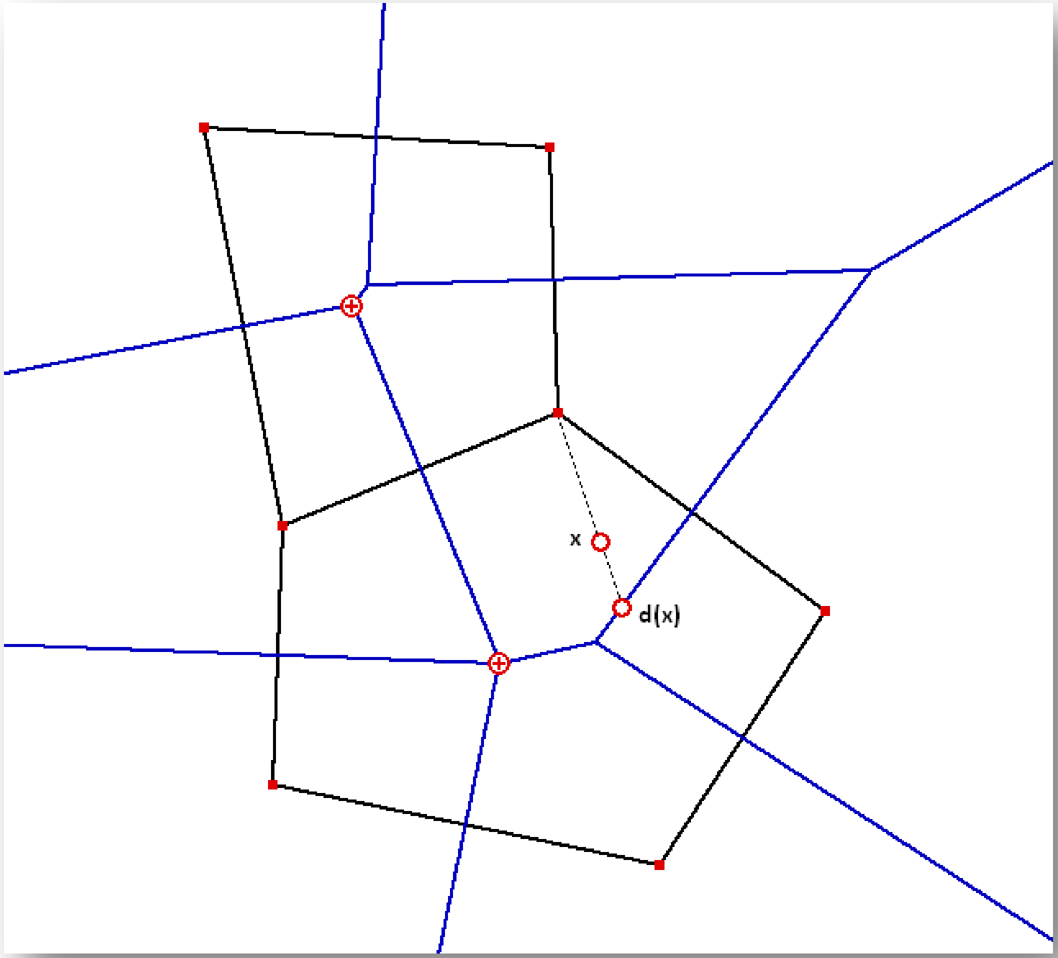
All critical points



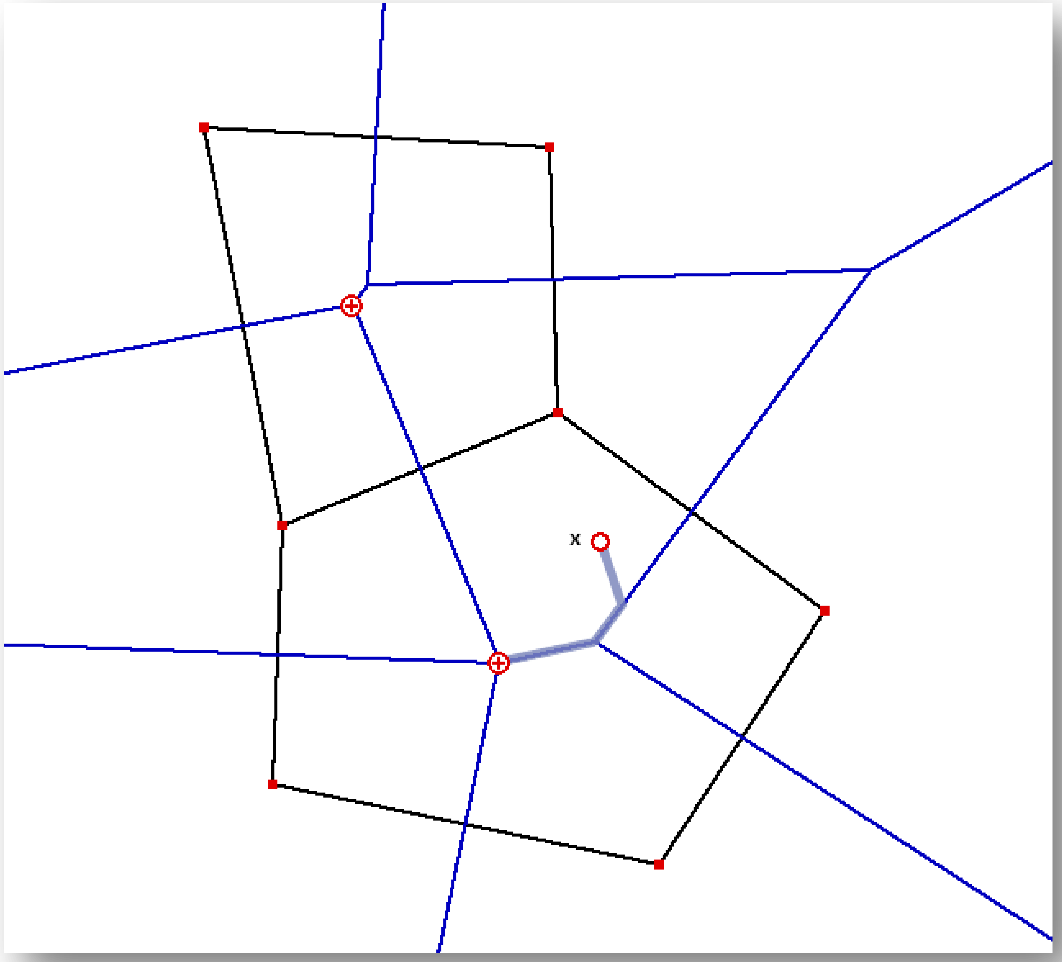
Maxima of h(x)



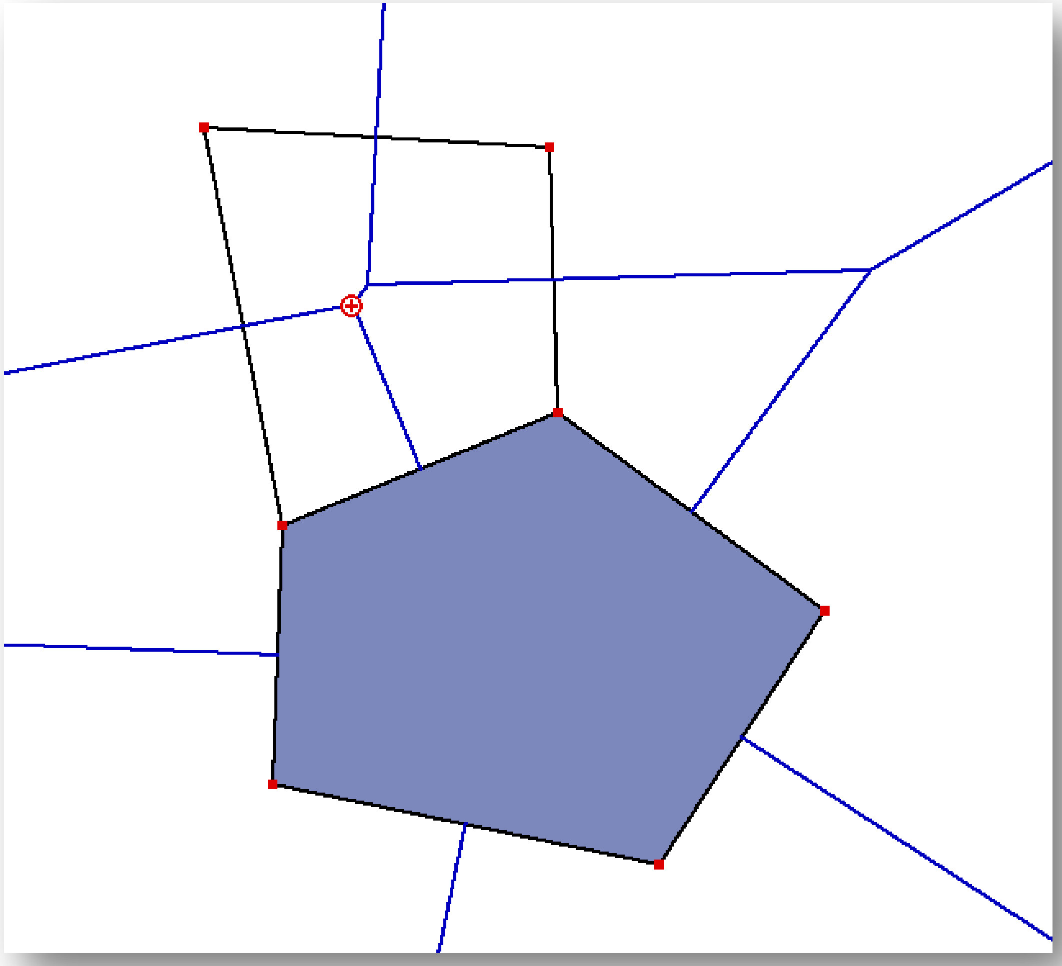
Driver of x



Orbit of x



Stable manifold



Flow complex