Spectral Surface Reconstruction

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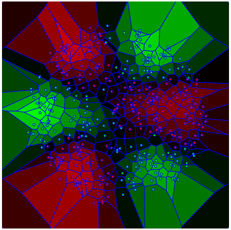
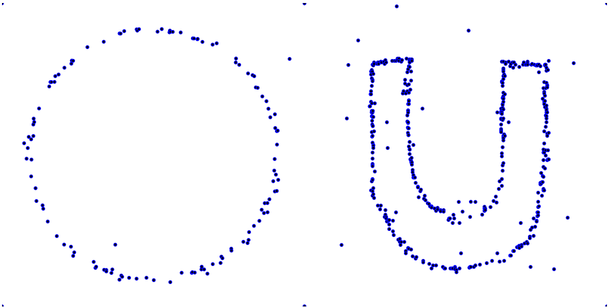


Outline

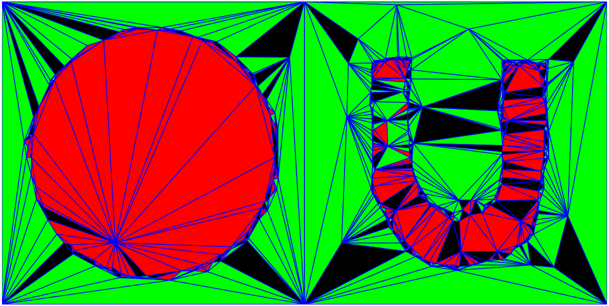
Motivation: Reconstruction

- Surface → cloud of sample points → watertight approximation
- Robust? Noise, outliers (laser scanner!), holes.
- Geom, top.



The EigenCrust Algorithm

- Geometric heuristics
- Transcends local problems by taking a global view
- *No holes* even in the presence of *noise* and *unsampled patches*

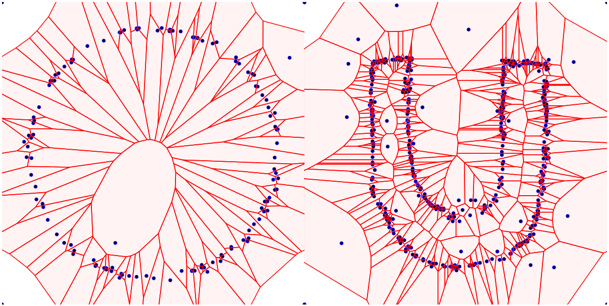


What is a Surface?

- Codimension 1 submanifold of ambient space
- No intersections, no boundary, manifold
- Surface = boundary of a *volume*.
- Search for manifold $\rightarrow$ automatically watertight!

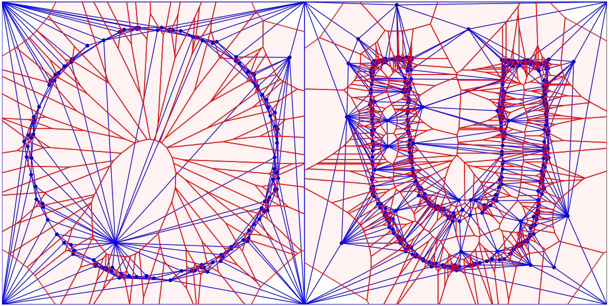
Voronoi / Delaunay (1)

- Voronoi cell
- Starting point: Spatial closeness



Voronoi / Delaunay (2)

- Delaunay duals Voronoi.

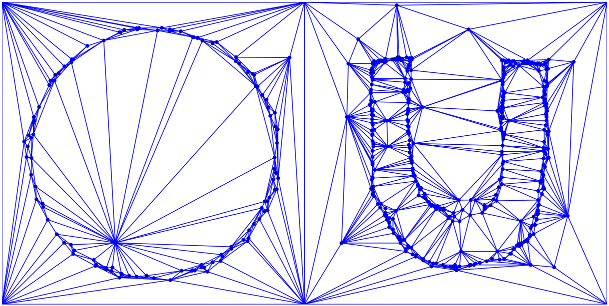


Starting Point: Delaunay Contains Surface

- Triangulation *contains* surface approximation → good starting point

Starting Point: Space partitioning

- Triangulation partitions space
- Label the tetrahedra $\rightarrow$ inside and outside
- Surface $=$ boundary $inside|outside$.



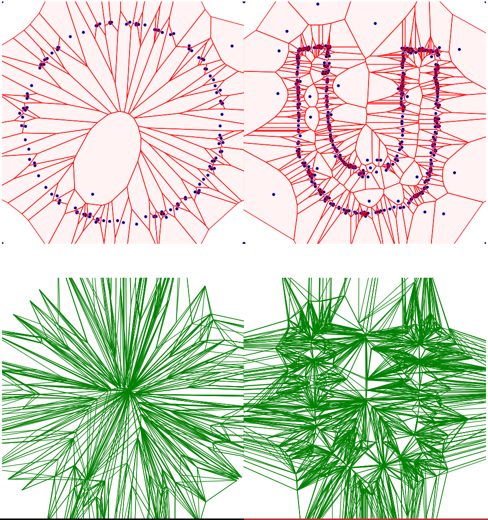
Skeleton

- Medial axis $\approx$ skeleton
- Deforms to surface's ambient complement
- (homotopy & homeomorphism!)

Voronoi Poles

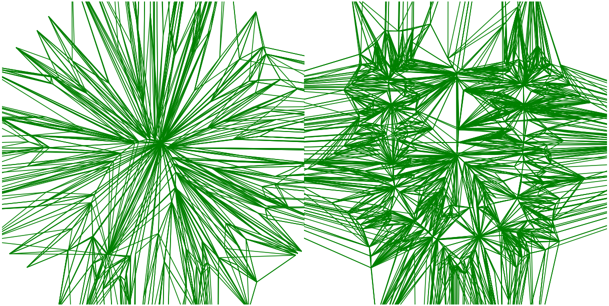
- Denser sampling $\rightarrow$ elongated cells
- Pole p^+ = furthest vertex of cell
- Pole p^-: only if $angle > \frac{\pi}{2}$
- Convergence to Medial Axis in 2D
- In 3D, "Surface" tetrahedra occur

Poles $\leftrightarrow$ Skeleton



Outline

- Delaunay triangulation is a combinatorial object (graph)
- So is its dual
- Good for algorithms!

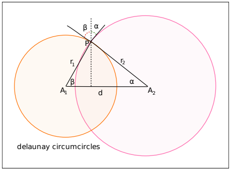


EigenCrust proper

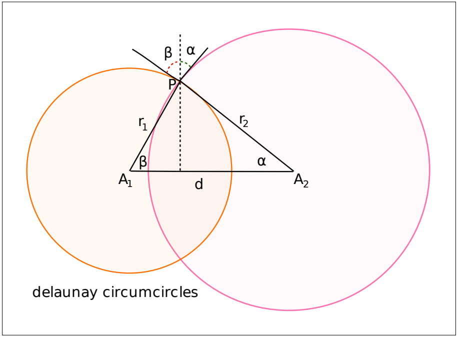
- Augment point cloud with bounding box
- Form pole graph (V, E, w):
- Poles belonging to a single vertex
- Poles of delaunay-neighboring vertices
- Edge weights: Geometrical Heuristic (sorry).
- Partition the pole graph
- Unlabel *suspicious tetrahedra* and re-partition.

EigenCrust (2)

- True MAT goes off into infinity → bounding box
- Authors use *negative* weights to great effect
- Weight: $-e^{4+4\cos\phi} - e^{4-4\cos\phi}$
- (unproven) justification: "Angle between circumspheres"

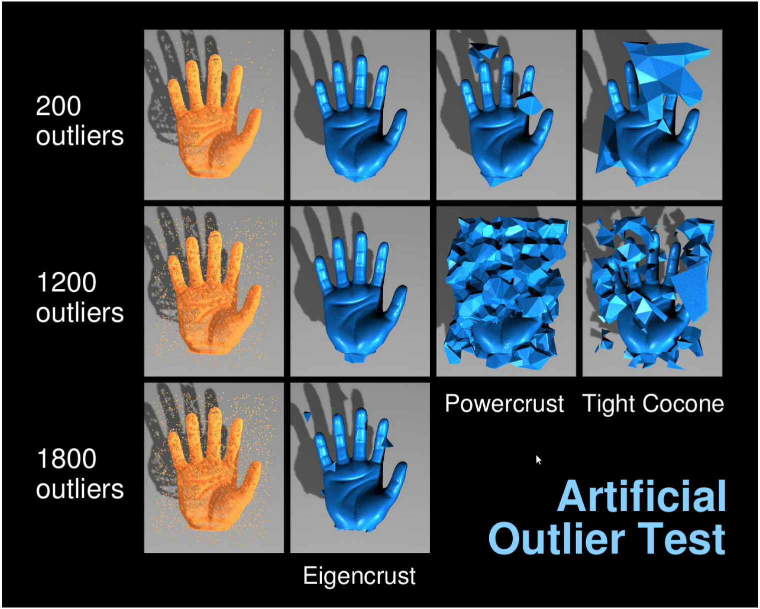


EigenCrust ($2\frac{1}{2}$)



EigenCrust (3)

- A priori OUTSIDE / INSIDE supernodes.
- Second step for non-poles / ambiguous.
- Next: a comparison, made by the authors


200 outliers
1200 outliers
1800 outliers
Powercrust
Tight Cocone
Eigencrust
Artificial Outlier Test

Outline

Before we proceed ...

We are going to need some seemingly unrelated stuff.
Please bear with me.

Finite-Dimensional Vector Spaces

- Recall the vector space axioms
- Linear transformation
- Basis
- Matrix
- Square matrix

We are talking ... Hilbert Spaces!

- Inner product: distances, angles
- $f \cdot g = \int_0^1 f(x)g(x)dx$
- Importance of linear operators
- Importance of hermitean operators

Operators

- Laplacian on $\mathbb{R}^n$ as second derivative vector
- It frequently appears in physics
- It is a linear operator.
- You already know its eigenvectors!

Harmonics

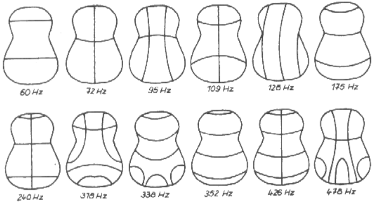
- Eigenfunctions of the Laplacian = Harmonics
- Harmonic Analysis is often a good idea
- Depends on domain
- Demo!

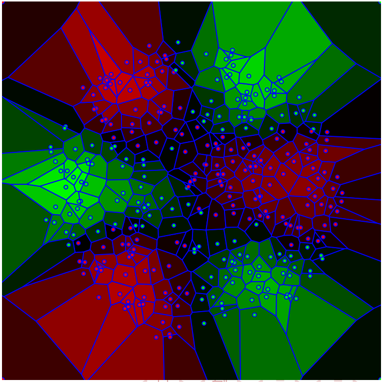
Natural Modes of Vibration

- Consider some solid object
- Tap it, it sounds
- You are hearing its spectrum!
- Normal Mode $\leftrightarrow$ Eigenvalue $\leftrightarrow$ Frequency $\leftrightarrow$ Energy (Why?)
- Can even be made visible
- Degeneracies

Harmonics & Eigenmodes

- Vibrations: Boundary value problem; but also ...
- finite model
- Resonances





$\mathbb{R}^n$ doesn't fit in my computer!

- Basic finite difference approximation
- Square grid
- Convergence
- generalizes to arbitrary grids & graphs

Self-Adjointness or Why the Spectrum is Real

- A desirable property: $\langle Ax, y \rangle = \langle x, Ay \rangle$
- Corresponds to *symmetric* matrix
- Real spectrum and *orthogonal* set of eigenvectors.
- Find $A = U \Lambda U^*$ (U rotation). Often miraculous!

Introducing Normalized Cuts

- Flexible formalism
- Segmentation by graph cuts
- Good, globally consistent solution
- Graph Theory $\leftrightarrow$ Linear Algebra
- Combinatorics $\leftrightarrow$ Numeric Methods
- Weighted, undirected graphs.

Combinatorial Laplacian / Graph Matrices

- Combinatorial Laplacian
- Affinity (generalized adjacency) matrix
  $$A_{ij} = \begin{cases} w_{ij} & \text{if } (i,j) \in E \\ 0 & \text{else} \end{cases}$$
- Degree matrix $\underline{D}$, diagonal D_{ii} = degree of vertex i
- Graph Laplacian $\underline{L} = \underline{D} - \underline{A}$
- Degree-normalized: $\underline{W} = \underline{D}^{-\frac{1}{2}} \underline{LD}^{-\frac{1}{2}}$ (transform vectors as needed)
- $\underline{W}$ remains sparse.

Outline of the NCuts Algorithm

- Construct a graph with weighted edges.
- Connectivity and Weights = adapt to model
- Partition along nodal sets
- Corresponding to λ_2
- (lowest is trivial)
- Object splits naturally – tightly connected parts vibrate together

Outline

Sparsely connected graphs → sparse matrices

- Small eigenvalue problems can be solved by direct methods (matrix factorizations)
- Prohibitive for large problems.
- Sparse matrices are made of zeroes ... mainly
- Matrix-Vector multiplication tends to be inexpensive
- Iterative methods very welcome.

Selective calculation of eigenvectors

- Calculating eigenpairs in $O(n)$ time per iteration
- Typical number of iterations $O(\sqrt{n})$
- But varies according to eigenstructure
- Positive definite; $\frac{x^T A x}{x^T x}$
- Get lowest first

More Applications

- Simulation (physics)
- Data Clustering: importance-weighted criterion
- / Text Mining
- Transductive Learning (global view)
- ...

Questions for you

- Proof of properties (reconstruction quality)?
- Incremental computations?
- Sharp corners $\longrightarrow$ hybrid
- ...

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